Relational Contracts, Limited Liability, and Employment Dynamics*

Yuk-fai Fong
Hong Kong University
of Science and Technology
yfong@ust.hk

Jin Li
Kellogg School of Management
Northwestern University
jin-li@kellogg.northwestern.edu

February 7, 2016

Abstract

This paper studies a relational contracting model in which the agent is protected by a limited liability constraint. The agent's effort is his private information and affects the output stochastically. We characterize the optimal relational contract and compare the dynamics of the relationship with that under the optimal long-term contract. Under the optimal relational contract, the relationship is less likely to survive, and the surviving relationship is less efficient. In addition, relationships always converge to a steady state under the optimal long-term contract, but they can cycle among different phases under the optimal relational contract.

JEL Classifications: C61, C73, J33, L24
Keywords: Relational Contracts, Limited Liability, Efficiency Wages

*A previous version of this paper has circulated under the title "Relational Contracts, Efficient Wages, and Employment Dynamics." We thank the editor and the anonymous referees for very detailed and helpful comments. We thank Guy Arie and Oscar Contreras for excellent research assistance. We especially thank Marina Halac for detailed comments. We also thank Ricardo Alonso, Attila Ambrus, Dan Barron, Dan Bernhardt, Bruno Biais, Odilon Camara, George Deltas, Willie Fuchs, Bob Gibbons, Michihiro Kandori, Francie Lafontaine, Bentley MacLeod, Tony Marino, Niko Matouschek, John Matsusaka, Thomas Marriotti, Arijit Mukherjee, Kevin Murphy, Marco Ottaviani, Mike Powell, Peter Schnabl, Michael Song, Jano Zabojnik and seminar participants at McGill, Michigan State, NUS, Northwestern, University of Delaware, University of Hong Kong, UIUC, USC, the 7th International Industrial Organization Conference, the Society of Labor Economists Annual Meeting 2009, the North American Summer Meeting of the Econometric Society 2009, the 8th annual CSIO-IDEI joint workshop on Industrial Organization, and the 2009 International Symposium on Contemporary Labor Economics for helpful comments and discussions. All remaining errors are ours.

1 Introduction

Many business relationships have three salient features. First, when banks lend to entrepreneurs, when manufacturers outsource to suppliers, and when firms hire workers, the relationships are often repeated. Second, actions chosen by one side of the relationship—the entrepreneur's investment, the supplier's quality, the worker's effort—may not be observed by the other side. Third, the side with the private actions has limited liability; entrepreneurs have limited wealth, suppliers have shallow pockets, and workers are protected by minimum wages.

The literature on dynamic contracting with limited liability has captured these features, and has been successful in providing insights into the dynamics of the relationship, with applications in corporate finance in particular; see Sannikov (2013) for a survey. A key assumption in this literature is that there are formal long-term agreements for the relationships, and more importantly, the principal can commit to these agreements.

However, in many situations—particularly employment relationships—the costs of drafting and enforcing long-term agreements can be prohibitively high. Consequently, the principal cannot be expected to commit to formal agreements. The relationship depends instead on relational contracts in which the parties keep their agreement because of their concerns for the loss of future payoffs. The main purpose of the paper is to investigate how the lack of commitment affects the dynamics of the relationship.

Specifically, we study a model of relational contracts with imperfect monitoring, which is an infinitely repeated principal-agent model where output is publicly observable but not contractible.[1] The agent privately chooses to work or shirk, and by working the agent increases the probability of the output being high. Unlike early models of dynamic moral hazard such as Lazear (1979), Shapiro and Stiglitz (1984), and Akerlof and Katz (1989), the output may be low even if the worker puts in effort. We model the limited liability constraint by requiring that the agent's pay each period not fall below an exogenously given wage floor, and we follow Levin (2003) by defining each relational contract as a Perfect Public Equilibrium (PPE) of the game. The optimal relational contract is the PPE that maximizes the principal's payoff.

We completely characterize the set of PPE payoffs through the method of Abreu, Pearce, Stacchetti (1990) and use this characterization to derive properties of the optimal relational contract. A profit-maximizing relationship begins with a "probation phase," during which the agent puts in effort and receives the lowest feasible wage regardless of output. If the output history has been sufficiently favorable, the worker transitions into the "reward phase," during which a high output leads to a wage above the wage floor. If the output history has been sufficiently unfavorable, the relationship transitions to the punishment phase. Depending on

[1] For a definitive treatment of relational contracts with observable actions, see MacLeod and Malcomson (1988).

the level of the wage floor, the agent will either be suspended, so that he is paid the wage floor but is not asked for effort, or the relationship will be terminated. These properties reflect the general lesson that reward should be backloaded in repeated interactions and are also featured in dynamic contracting models.

We then compare the optimal relational contract to a long-term contract with commitment. When the relationship does not have a high enough surplus or when players are not too patient, we show that non-commitment makes the relationship less likely to survive. Even if the relationship survives in the long run, non-commitment reduces its efficiency and lowers the agent's wage. Finally, whereas the relationship always converges to a steady state in the long run under long-term contracts, it can cycle among different phases under the relational contract.

One reason for the differences is that the amount of the surplus in the relationship constrains the reward the principal can give out when she cannot commit. As a result, the back-loading of the reward is incomplete when the surplus is low, so that the reward region is non-absorbing—low outputs will move the agent out of the reward region. In contrast, the reward region is absorbing under long-term contracts. The second reason is that lack of commitment can change the way the agent is punished. Since lack of commitment reduces the value of the relationship, punishment is more likely to take the form of termination rather than suspension. For some parameter ranges, the relationship survives with probability 1 under the optimal long-term contract, yet it is terminated with probability 1 under the optimal relational contract.

This paper contributes to the literature on relational incentive contracts, where neither the principal nor the agent can commit to the contract; see Malcomson (2013) for a survey. Within this literature, the closest paper is Hörner and Samuelson (Forthcoming), who also study a discrete-time model of repeated moral hazard with no commitment on both sides. In their model, a principal finances a project and chooses the scale of the project in each period. The choice of scale has a significant effect on the dynamics of the relationship. In particular, the relationship never terminates. Hörner and Samuelson (Forthcoming) also assume that the efficient outcome can be supported as a stage-game equilibrium, and consequently, the relationship becomes efficient in the long run with probability 1 in their model. In our model, in contrast, the efficient outcome cannot be supported as a stage-game equilibrium when the relationship has low surplus. In this case, inefficiency can occur in the relationship in the long run, and the relationship may also terminate.

This paper is also related to three other strands of literature that generate dynamics in relationships but make different assumptions about commitment and information structure. The first strand consists of repeated principal-agent models with full commitment; see for example, Green (1987), Spear and Srivastava (1987), and Thomas and Worrall (1990). These papers pioneer the use of recursive techniques to characterize the optimal long-term contract. The agent's expected utility is a sufficient statistic for the past history, and therefore, determines the future dynamics of the relationship. When there is full commitment, Thomas and Worrall (1990) show that the agent's expected utility becomes arbitrarily negative with probability 1. This property does not arise in our model both because the agent cannot commit (so that he can take his outside option) and because the agent has a limited liability constraint.

The second strand of related literature consists of repeated principal-agent models with one-sided commitment, so that the agent has the option of quitting the relationship in every period; see Biais, Mariotti, and Rochet (2013) and Sannikov (2013) for surveys.[2] The two papers that are most related to ours are Biais, Mariotti, Plantin, and Rochet (2004), and Biais, Mariotti, and Rochet (2013). Both include analyses of discrete-time models in which the agent is risk-neutral, has a limited liability constraint, and has an equal discount factor as the principal. As in our model, these papers show that the payoff frontier consists of a punishment-, a probationary-, and a reward region. In these papers, because the principal can commit, the reward region is absorbing. In contrast, the principal cannot commit in our model, and the reward region is non-absorbing when the relationship has low surplus.

Another closely related paper in this literature is Zhu (2013), who studies the repeated principal-agent problem in continuous-time.[3] As in our model, Zhu (2013) allows for the possibility of suspension to occur. Different from ours, Zhu (2013) assumes that the principal is more patient than the agent. This assumption implies that the reward region is never absorbing, and as a result, the relationship either survives with probability 0 or probability 1 in the long run. In contrast, the reward region is absorbing in our model when the relationship has high surplus. In this case, the relationship survives with a positive (but less than 1) probability.

The third strand of related literature consists of models with limited commitment and perfect information; see for example, Thomas and Worrall (1994), Ray (2002), Albuquerque and Hopenhayn (2004), and Thomas and Worrall (2010). As in our model, the lack of commitment generates dynamics in relationships. But because information is perfect, the relationships in these models become more efficient over time, and the reward regions in these models are absorbing. In contrast, because the agent's action is private in our model, his continuation payoff falls following a low output. When the relationship has low surplus, the reward region in our model is non-absorbing. As a result, the relationship may terminate. And even if the relationship does not terminate, the total surplus of the relationship fluctuates over time.

[2] Many papers in this literature assume that the agent has a limited liability constraint that is weakly greater than the agent's per period outside option. With this assumption, there is often no distinction whether the agent can or cannot quit the relationship; see for example, DeMarzo and Sannikov (2006) for a discussion of this implication.

[3] In Zhu (2013) and several other continuous-time repeated principal-agent models (DeMarzo and Sannikov (2006), Bias, Mariotti, Plantin, Rochet (2007), the output-generating process is continuous. In contrast, the continuous-time limit of the output generating process in our model is a Poisson process. The discontinuity in the output-generating process is necessary to fully capture the effect that the principal cannot commit.

Finally, our model is also related to the class of efficiency wage models because the worker has ex ante rents in the relationship and there is a wage floor. Our model adds to classic efficiency wage models such as Shapiro and Stiglitz (1984) by featuring a) stochastic production on the equilibrium path and b) an explicit modeling of the wage floor. The combination of these two features helps generate more realistic patterns of turnover and pay, which we discuss in Section 4.

The rest of the paper is organized as follows. We set up the model in Section 2. The PPE payoff set and the optimal relational contract are characterized in Section 3. In Section 4, we explore the implications of the model. Section 5 considers several extensions of the main model, and Section 6 concludes. The proofs of all the formal results are relegated to the Appendix.

2 Setup

There is one principal and one agent. Both are risk neutral, infinitely lived, and have a common discount factor δ. Time is discrete and indexed by $t \in \{1, 2, ..., \infty\}$.

At the beginning of each period t, the principal decides whether to offer a contract to the agent: $d_t^P \in \{0, 1\}$. If a contract is offered, it specifies a legally enforceable wage $w_t \geq \underline{w}$, where $\underline{w} \in R$ is an exogenously given wage floor.[4]

In many relational contracting models, the contract also includes a discretionary end-of-the-period bonus. The current setup is chosen to simplify notation and to facilitate the comparison with efficiency wage models. These two setups are equivalent in the sense that they give rise to the same set of equilibrium payoffs, and the results obtained here can be directly translated into a version with a discretionary bonus.[5] In our setup, the discretionary bonus can be thought of as being postponed until the beginning of the next period, and it becomes part of the wage offered. Specifically, for any wage $w_{t+1} > \underline{w}$, it is equivalent that the principal pays out a bonus equal to $(w_{t+1} - \underline{w})/\delta$ at the end of period t and a wage equal to $\underline{w}$ at the beginning of period $t + 1$. In our discussion below, we will refer to wage payments above the wage floor as bonus payments.

If the principal offers a contract, the agent decides whether to accept or reject it: $d_t^A \in \{0, 1\}$. If he accepts the offer, the principal pays out wage w_t. The agent then chooses effort $e_t \in \{0, 1\}$, and output $Y_t \in \{0, y\}$ is realized. If the agent works ($e_t = 1$), he incurs a cost of effort c, and Y_t is equal to y with probability $p \in (0, 1)$ and 0 with probability $1 - p$. If the agent shirks

[4] Notice that the limited-liability constraint is not a one-off constraint, but is required to hold period-by-period.
[5] MacLeod and Malcomson (1998) provide a formal proof for a model with symmetric information. Their proof can be adapted to our model.

($e_t = 0$), no effort cost is incurred, and Y_t is equal to y with probability $q < p$. The agent's effort choice is his private information. Output is publicly observed.

If the principal does not make a contract offer ($d_t^P = 0$) or if the agent rejects an offer ($d_t^A = 0$), then the players receive their outside options for the period. Let the agent's per-period outside option be $\underline{u}$ and that of the principal be $\underline{v}$. At the end of the period, both the principal and the agent observe the realization of a random variable x_t that is uniformly distributed between 0 and 1, and we also assume that a random variable x_0 is realized at the very beginning of the game.[6] To make the analysis interesting, we assume that the value of the relationship exceeds the sum of outside options if and only the agent puts in effort: $py - c > \underline{u} + \underline{v} > qy$. We summarize the timing of the stage-game in Figure 1 below.

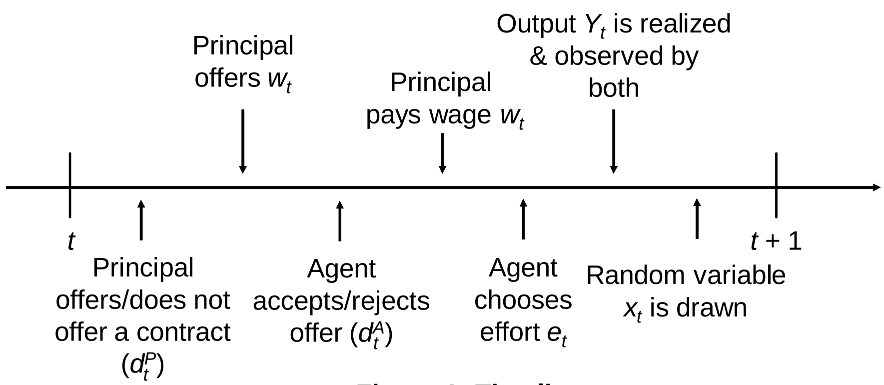

Figure 1: Timeline

The stage-game is repeated infinitely. At the beginning of any period t, the expected payoffs to the principal and the agent are given by

$$v_t = (1-\delta)\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}[d_t^P d_t^A\left(y(q+(p-q)e_\tau)-w_\tau\right)+\left(1-d_t^P d_t^A\right)\underline{v}]\right];$$
$$u_t = (1-\delta)\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}[d_t^P d_t^A\left(w_\tau - ce_\tau\right)+\left(1-d_t^P d_t^A\right)\underline{u}]\right],$$

where we multiply throughout by $(1-\delta)$ to express the payoffs as per period averages.

The solution concept of the game is perfect public equilibrium (PPE). A PPE is a strategy profile such that a) players choose actions that depend on the public history, and b) the strategy profiles following any public history form a Nash Equilibrium of the game following that history. In particular, this implies that the agent's action choice does not depend on his past effort choices. This restriction is without loss of generality, since the information about the principal's

[6] The public randomization device is a commonly made assumption in models of repeated games to convexify the equilibrium payoffs; see, for example, Mailath and Samuelson (2006, Section 3.4) for a discussion of its roles.

actions is public, so our model is a game of imperfect monitoring with "product structure." In this case, the agent does not gain by conditioning his actions on past private effort choices; see Fudenberg and Levine (1994).

Formally, denote $h_t = \{d_t^P, w_t, d_t^A, y_t, x_t\}$ as the public events that occur in period t. Let $h^t = \{h_n\}_{n=1}^{t-1}$ be a public history path at the beginning of period t, and $h^1 = \{x_0\}$. Let $H^t = \{h^t\}$ be the set of public history paths until time t, and define $H = \cup_t H^t$ as the set of public histories. In period t, depending on the set of public history h^t, the principal decides whether to offer a contract to the agent $\left(d_t^P\right)$, and if so, what wage to offer (w_t). The agent then decides whether to accept the principal's contract $\left(d_t^A\right)$, and if he does, what effort level to choose (e_t). Denote the principal's strategy by s^P and the agent's public strategy by s^A. Let $v(s^P, s^A|h_t)$ and $u(s^P, s^A|h_t)$ be the principal's and the agent's expected payoffs following public history h_t. A strategy profile (s^P, s^A) is a PPE when for each public history h_t,

$$\begin{aligned} s^P &\in \arg\max_{\widetilde{s}^P} v(\widetilde{s}^P, s^A|h_t), \\ s^A &\in \arg\max_{\widetilde{s}^A} u(s^P, \widetilde{s}^A|h_t). \end{aligned}$$

We denote each PPE as a *relational contract*. The *optimal relational contract* is the PPE that maximizes the principal's payoff at the beginning of the first period.

We solve for the optimal relational contract in the next section. To make the analysis interesting, we assume the following two conditions are satisfied. The first condition implies that the wage floor is sufficiently high so the limited liability condition matters:

$$\underline{w} > \underline{u} - \frac{qc}{p-q}. \tag{LLB}$$

Otherwise, the principal is able to set sufficiently low wages to extract all of the surplus in the relationship, and in this case, the optimal relational contract is essentially stationary; see Levin (2003). The second condition guarantees a nontrivial relational contract exists:

$$py - c - \underline{u} - \underline{v} > \max\{\frac{1-p\delta}{\delta}\frac{c}{p-q}, \underline{w} - c - \underline{u}\}. \tag{NT}$$

Specifically, this condition guarantees the following relational contract can be supported. In each period, the agent puts in effort. If the output is low, the relationship is terminated. Otherwise, the agent is given a reward (to be paid out at the beginning of next period). The first inequality (that the left hand side is bigger than $(1-p\delta)\, c/\left(\delta\left(p-q\right)\right)$ ensures that there exists a large enough reward to induce the agent to put in effort. The second inequality ensures that the principal finds it incentive compatible to pay the reward.

3 Analysis

In this section, we characterize the PPE payoff set using the technique developed by Abreu, Pearce, and Stacchetti (1990). The first subsection gives a recursive representation of the PPE payoff set. The second subsection characterizes the PPE payoff frontier and describes the optimal relational contract.

3.1 PPE Payoff Set

To give a recursive representation of the PPE payoffs, we start by establishing conditions a PPE payoff must satisfy. Denote the set of PPE payoffs by $\mathcal{E}$. For payoffs in $\mathcal{E}$, the recursive representation consists of an action profile in the current period, and continuation payoffs in $\mathcal{E}$ that depend on publicly observable outcomes. In our setting, it is without loss of generality to assume that if a publicly observable deviation occurs, the parties terminate their relationship, as this gives each party its minmax payoff. This implies that only the description of on-path actions and continuation payoffs is required.

Consider a PPE payoff pair $(u, v) \in \mathcal{E}$. The possible action profiles that *supports* (u, v) include effort, suspension (no effort), and exit. For (u, v) to be supported with effort, the equilibrium specifies a wage payment w, the agent's and the principal's continuation payoffs following a low output (u_l, v_l), and those following a high output (u_h, v_h). In addition, the following conditions must hold: first, the wages must satisfy the limited-liability condition:

$$w \geq \underline{w}; \tag{LL}$$

second, the agent must be willing to exert effort:

$$\delta (p - q) (u_h - u_l) \geq (1 - \delta) c; \tag{IC}$$

third, both parties' payoffs must equal the weighted sum of current and future payoffs:

$$v = (1 - \delta) (py - w) + \delta (pv_h + (1 - p) v_l), \tag{PK$_\mathrm{P}$}$$

$$u = (1 - \delta) (w - c) + \delta (pu_h + (1 - p) u_l); \tag{PK$_\mathrm{A}$}$$

and, finally, the continuation payoffs must be in the set of PPE payoffs:

$$(u_l, v_l) \in \mathcal{E}, \tag{SE$_\mathrm{l}$}$$

$$(u_h, v_h) \in \mathcal{E}. \tag{SE$_\mathrm{h}$}$$

The constraints (SE$_\mathrm{l}$) and (SE$_\mathrm{h}$) correspond to the non-reneging constraints in models with bonus payments. Here, the reward for the agent takes a form of higher wage next period, and the non-reneging constraints then restrict the continuation payoffs to be in $\mathcal{E}$.

Next, suppose (u, v) is supported with suspension. The equilibrium specifies a wage payment w and continuation payoff (u_s, v_s), where the subscript s denotes that the agent is suspended from working this period. Because the agent is asked not to work, his incentive constraint (IC) is irrelevant. The promise-keeping constraints are given by

$$\begin{aligned} u &= (1-\delta)\, w + \delta u_s, \\ v &= (1-\delta)\,(qy - w) + \delta v_s, \end{aligned}$$

and the self-enforcing constraint requires $(u_s, v_s) \in \mathcal{E}$.

Third, suppose (u, v) is supported with exit. The parties take their outside options for one period, and their continuation payoffs are given by (u_x, v_x). To support (u, v) with exit, (LL) and (IC) are irrelevant. The promise-keeping constraints give

$$\begin{aligned} u &= (1-\delta)\, \underline{u} + \delta u_x, \\ v &= (1-\delta)\, \underline{v} + \delta v_x, \end{aligned}$$

and the self-enforcing constraints require $(u_x, v_x) \in \mathcal{E}$.

Finally, (u, v) can be supported with randomization, so it is a convex combination of payoffs in $\mathcal{E}$, using the public randomization device.

To characterize $\mathcal{E}$, we need to determine, for each $(u, v) \in \mathcal{E}$, whether it is supported by effort, suspension, exit, or randomization. We then need to specify the corresponding actions and continuation payoffs, and in the case of randomization, how (u, v) is randomized. We now characterize the PPE payoff set and describe the optimal relational contract.

3.2 PPE Payoff Frontier and Optimal Relational Contract

Define the payoff frontier as

$$f(u) \equiv \max\{v' : (u, v') \in \mathcal{E}\}.$$

In other words, $f(u)$ is the principal's highest PPE payoff when the agent's payoff is u. Let $\overline{u} = \max\{u : (u, v) \in \mathcal{E}\}$ be the largest PPE payoff of the agent. It is clear that $\mathcal{E}$ is compact, so f and $\overline{u}$ are well defined, and furthermore, f is concave since a public randomization device is available. For any payoff $(u, f(u))$ on the frontier, the continuation payoffs must remain on the frontier. This is because the principal's actions are publicly observable, so she is never punished on the equilibrium path.[7] As a result, the PPE payoff set and the optimal relational contract are completely determined by the payoff frontier $f(u)$.

[7] When multiple parties take private actions, joint punishments may be necessary; see, for example, Green and Porter (1984), Athey and Bagwell (2001), and the second part of Levin (2003).

To characterize the PPE payoff frontier, we first need to determine, for each payoff pair $(u, f(u))$, whether it is supported by randomization or by a pure action, and if it is supported by a pure action, whether the action is exit, suspension, or effort. Finally, if it is supported by suspension or effort, we need to determine the agent's wage level.

To simplify the exposition, let $k \equiv (1-\delta)c/(\delta(p-q))$ be the smallest difference between the agent's continuation payoffs following a high and low output $(u_h - u_l)$ so that (IC) binds. Also let $L(u) \equiv u/\delta - (1-\delta)(\underline{w} + qc/(p-q))/\delta$ equal the agent's continuation payoff following low output if he is paid $\underline{w}$ and (IC) binds. In this case, the agent's continuation payoff following high output is then given by $L(u)+k$. Finally, let $\delta^* \equiv 1/(1-p+(p-q)(py-\underline{w}-\underline{v})/c)$ equal the cutoff discount factor above which first best can be attained by a relational contract. Our first result describes the PPE payoff frontier.

LEMMA 1: *There exist a u_1^* and a u_2^* that divide the payoff frontier into three regions.*
(i.) (Punishment Region) For all $u \in (\underline{u}, u_1^*)$, $(u, f(u))$ can be supported with randomization between $(\underline{u}, f(\underline{u}))$ and $(u_1^*, f(u_1^*))$. There exists $w^* < \underline{u}$ such that, if $\underline{w} < w^*$, $(\underline{u}, f(\underline{u}))$ is supported with suspension and the agent is paid $\underline{w}$, and if $\underline{w} \geq w^*$, $(\underline{u}, f(\underline{u}))$ is supported with exit.
(ii.) (Probationary Region) For all $u \in [u_1^*, u_2^*]$, $(u, f(u))$ can be supported with effort and $w(u) = \underline{w}$.
(iii.) (Reward Region) For $u \in (u_2^*, \overline{u}]$, $f'(u) = -1$. $(u, f(u))$ can be supported with effort and $w(u) = \underline{w} + (u - u_2^*)/(1-\delta)$.
(iv.)

$$u_1^* = \begin{cases} L^{-1}(\underline{u}) & \text{if } \underline{w} < \underline{u} + (1-\delta q)c/(p-q) \\ \underline{w} - (1-q)c/(p-q) & \text{if } \underline{w} \geq \underline{u} + (1-\delta q)c/(p-q) \end{cases}.$$

(v.)

$$u_2^* = \begin{cases} L^{-1}(\overline{u} - k) & \text{if } \delta < \delta^* \\ \underline{w} + qc/(p-q) & \text{if } \delta \geq \delta^* \end{cases}.$$

Lemma 1 shows that the payoff frontier consists of a punishment region, a probationary region, and a reward region. In the reward region, the agent puts in effort and receives wage above $\underline{w}$. In the probationary region, the agent puts in effort but his wage is always $\underline{w}$. In the punishment region, the agent does not put in effort and inefficiency occurs. The three regions arise because of the familiar logic of back-loading of the reward to the agent. Essentially, by back-loading the reward and delaying the punishment, the principal reduces the payment to the agent by allowing the incentives to be reused. This three-region structure of the payoff frontier is a common feature of dynamic-contracting models with limited liability; see for example, DeMarzo and Fishman (2007) and Biais, Mariotti, Plantin, and Rochet (2007).

While the three-region structure is common in the literature, our payoff frontier has two notable features. The first feature concerns with the use of suspension in the punishment region. Part (i) of the lemma also shows that suspension arises if and only if the wage floor $\underline{w}$ is below a cutoff w^*, where $w^* \leq \underline{u}$; see Figure 2a and 2b below for illustrations. To see what determines the use of suspension or exit, note that when suspension is used, the joint (contemporaneous) payoff of the principal and the agent is qy, which is smaller than $\underline{u} + \underline{v}$, the joint payoff when exit is used. Suspension therefore creates a larger efficiency loss, and, everything else equal, is the less preferred method of punishment. However, when $\underline{w} < \underline{u}$, suspension has the advantage of allowing the principal to more harshly punish the agent since it gives the agent a payoff lower than his outside option. As $\underline{w}$ decreases, suspension becomes a more effective tool for punishment, and eventually dominates exit.

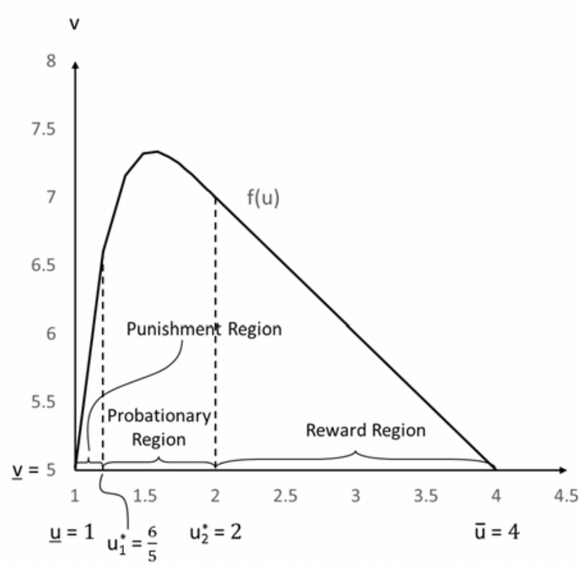

Figure 2a: PPE Payoff Frontier with Termination

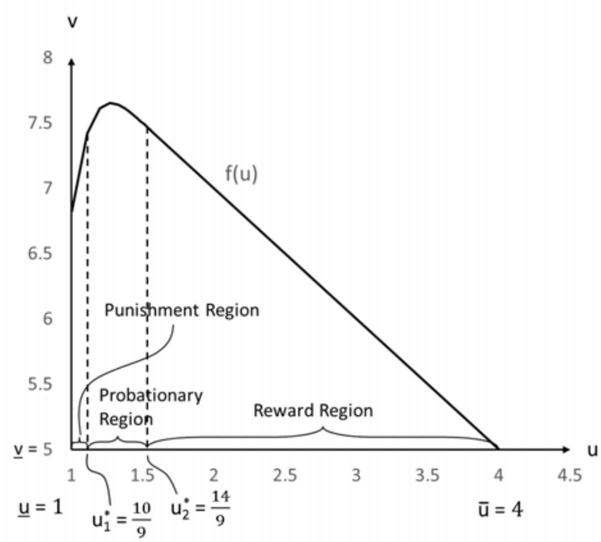

Figure 2b: PPE Payoff Frontier with Suspension

The figures are generated based on these parameters: $p = 1/2$, $q = 1/4$, $\delta = 0.8$, $c = 1$, $y = 20$, $\underline{u} = 1$, and $\underline{v} = 5$. The only difference between them is that in Figure 2a, $\underline{w} = 1$ and in Figure 2b, $\underline{w} = 5/9$.

The use of suspension contrasts with existing dynamic-contracting models in which the principal and the agent have equal discount factors; see for example, Biais, Mariotti, Plantin, and Rochet (2004) and Biais, Mariotti, Rochet (2013). There, the analyses restrict to contracts in which that the agent puts in effort as long as the relationship survives, and assumptions on parameters are made to ensure that the restriction is valid. In particular, $\underline{w} = \underline{u} = 0$ in these models. In contrast, we allow for $\underline{w} < \underline{u}$, and therefore, suspension can occur. When the principal is more patient than the agent, Zhu (2013) shows that suspension can occur for a reason similar to ours. In Zhu (2013), however, suspension occurs when the agent's continuation payoff reaches a threshold that is typically greater than the agent's outside option. In contrast, suspension occurs only at $\underline{u}$ in our model. The difference arises in part because Zhu (2013) considers a continuous-time model, so there is no randomization region in the payoff frontier in his model. In addition, because the principal is more patient than the agent, there is gain in front-loading the reward to the agent, and relatedly, gain in front-loading the punishment. Consequently, suspension can occur even though the agent's payoff is above his outside option.

Another key feature of the payoff frontier is concerned with the property of the reward region. Part (v) of the lemma shows that the reward region's left boundary, u_2^*, depends on the surplus of the relationship. When the relationship has *high surplus* ($\delta \geq \delta^*$), u_2^* satisfies $u_l(u_2^*) = u_2^* = \underline{w} + qc/(p-q)$, so that at u_2^*, the agent's continuation payoff stays at u_2^* even if output is low, and therefore, the reward region is absorbing. But when the relationship has *low surplus* ($\delta < \delta^*$), u_2^* is given by $u_h(u_2^*) = \overline{u}$ (so that the agent's continuation payoff following a high output is at the maximal equilibrium level). In this case, $u_l(u_2^*) < u_2^* < \underline{w} + qc/(p-q)$, so the agent's continuation payoff following a low output falls below u_2^*, and the reward region is non-absorbing.

The non-absorbing property of the reward region contrasts with existing models in which the principal and the agent have equal discount factors and the principal can commit (Biais, Mariotti, Plantin, and Rochet (2004) and Biais, Mariotti, Rochet (2013)). In these models, the reward region is always absorbing. The difference arises precisely because the principal cannot commit in our model. The lack of commitment implies that the principal's payoff in the relationship cannot be lower than her outside option, and as a result, her reward to the agent is bounded by the surplus of the relationship. When the surplus of the relationship is low, the principal cannot delay rewarding the agent until his continuation payoff reaches $\underline{w} + qc/(p-q)$, as she would if she could commit. Doing so would require the principal to eventually pay a reward larger than the surplus of the relationship, at which point she would renege. The principal must instead pay smaller rewards earlier in the relationship, which implies that the agent must be punished following low output in the sense that his continuation payoff falls even when his payoff is in the reward region. This is why the reward region is non-absorbing when the surplus is low and the principal cannot commit.

Finally, part (iv) shows that depending on the wage floor, there are two ways of determining u_1^*, the right boundary of the punishment region. Part (iv) shows that when $\underline{w} \leq \underline{u} + (1-\delta q)\, c/(p-q)$, u_1^* satisfies $u_l(u_1^*) = L(u_1^*) = \underline{u}$. In other words, u_1^* is the agent's smallest equilibrium payoff such that effort is feasible (since if $e = 1$ for any $u < u_1^*$, the agent's continuation payoff following a low output would have to fall below $\underline{u}$ to motivate the agent). In this case, the agent's continuation payoff always increases when output is high. When $\underline{w} > \underline{u} + (1-\delta q)\, c/(p-q)$, however, the agent's continuation payoff may decrease even if output is high, i.e., $u_h(u) < u$. We show in the proof in the appendix that, when $u_h(u) < u$, the agent's payoff must belong to the punishment region, and u_1^* is then determined by $u_h(u_1^*) = u_1^* = \underline{w} - (1-q)c/(p-q)$. In this case, once the agent's payoff falls (weakly) below u_1^*, his continuation payoff will never exceed u_1^*, so the punishment region is absorbing. Unlike the reward region, whether the punishment region is absorbing or not has no effect on the long-run dynamics of the relationship, as we will see below.

Next, Proposition 1 uses the characterization of the payoff frontier in Lemma 1 to describe the optimal relational contract. Under the optimal relational contract, the principal's payoff is given by $\max_{u\in[\underline{u},\overline{u}]} f(u)$.

PROPOSITION 1: *The optimal relation contract satisfies the following:*

First period: the agent's payoff satisfies $u \in [u_1^*, u_2^*]$. *The agent receives* $w = \underline{w}$, *and chooses effort. If output is high, the agent's continuation payoff increases, and it falls otherwise.*

Subsequent periods: The agent's expected payoff is given by $u \in \{\underline{u}\} \cup [u_1^*, u_2^*] \cup (u_2^*, \overline{u}]$.

(i.) If $u = \underline{u}$, *the agent is suspended if* $\underline{w} < w^*$, *and the agent exits otherwise.*

(ii.) If $u \in [u_1^*, u_2^*]$, *the agent receives* $w = \underline{w}$, *and chooses effort. His continuation payoffs are given by* $u_l(u) = L(u) \leq u$ *and* $u_h(u) = L(u) + k \geq u$.

(iii.) If $u \in (u_2^*, \overline{u}]$, *the agent receives* $w(u) = \underline{w} + (u - u_2^*)/(1-\delta)$, *and chooses effort. His continuation payoffs are given by* $u_l(u) = L(u_2^*)$ *and* $u_h(u) = L(u_2^*) + k$. *The choice of the continuation payoffs are unique when* $\delta < \delta^*$.

Proposition 1 shows that under the optimal relational contract, the relationship starts in the probationary region, which implies that the total surplus is not maximized. The inefficiency of the optimal relationship follows from the familiar trade-off between efficiency enhancement and rent extraction. By lowering the total surplus, the principal can extract more rents from the agent and therefore increase her payoff. Proposition 1 also shows that, when the agent is asked to put in effort, his IC constraint holds with equality ($u_h - u_l = k$). This is also a standard result and follows directly from the concavity of f. Part (iii.) shows that, if $\delta < \delta^*$ and $u \in (u_2^*, \overline{u}]$, the choices of u_l and u_h are unique. In general, however, their choices are not unique, and there might be multiple ways to implement the optimal relational contract. This can occur, in particular, when both u_l and u_h are in the interiors of linear regions of the payoff frontier. But regardless of the implementations used, they result in the same dynamics of the relationship and the same long-run outcomes.

Proposition 1 has a number of implications on the agent's pay and turnover dynamics. It implies that the pay of the worker is initially low and is insensitive to output. Since the relationship starts in the probationary region, the pay of the worker starts at $\underline{w}$. As long as the agent's payoff remains in the probationary region, his pay is always equal to $\underline{w}$, and therefore, is independent of output. While output does not affect the agent's pay in the short run, it affects the agent's continuation payoff. A high output increases the agent's continuation payoff, and when high outputs occur sufficiently often, the agent's continuation payoff moves to the reward region. In the reward region, the pay of the agent is higher than $\underline{w}$, and output affects the agent's pay next period.

Proposition 1 also implies that if termination occurs in the relationship, the rate of termination may be inverse-U shaped with respect to the duration of the relationship. In particular, suppose $\underline{w} \geq w^*$ so that punishment takes the form of exit. Part (i) of Proposition 1 then implies that the relationship terminates as soon as the agent exits, since once the agent exits, his continuation payoff remains at $\underline{u}$, and he will exit next period as well. Recall that the relationship starts in the probationary region, so the principal does not terminate the agent immediately for a low output. Instead, she lowers the agent's continuation payoff, and therefore, raises the agent's termination probability in the future. This explains why the probability of termination is low at the beginning of each relationship. But in the long run, the termination rate converges to zero when the relationship has high surplus. Notice that when $\delta \geq \delta^*$, $[u_2^*, \overline{u}]$ is absorbing since $u_l(u_2^*) = u_2^*$. Once the agent's payoff exceeds u_2^*, the relationship therefore never terminates. However, as long as the agent's continuation payoff has not reached $[u_2^*, \overline{u}]$, he always faces the risk that he will eventually be terminated. As time passes by, the agent is either terminated or reaches $[u_2^*, \overline{u}]$. Longer relationships are more likely to have reached $[u_2^*, \overline{u}]$, so that the termination rate eventually converges to zero. Since the termination rate is also low at the beginning of the relationship, this suggests that the termination rate can be inverse-U shaped.

Next, Corollary 1 describes the long-run outcomes of the relationship.

COROLLARY 1: *The following holds.*

(i.) High Surplus and High Wage Floor ($\delta \geq \delta^*$, $\underline{w} \geq w^*$). The relationship either terminates or stays in the reward region: $\lim_{t\to\infty} \Pr(u_t \geq u_2^*) = 1 - \lim_{t\to\infty} \Pr(u_t = \underline{u}) \in (0,1)$.

(ii.) High Surplus and Low Wage Floor ($\delta \geq \delta^*$, $\underline{w} < w^*$). The relationship ends in the reward region with probability 1: $\lim_{t\to\infty} \Pr(u_t \geq u_2^*) = 1$.

(iii.) Low Surplus and High Wage Floor ($\delta < \delta^*$, $\underline{w} \geq w^*$). The relationship terminates with probability 1: $\lim_{t\to\infty} \Pr(u_t = \underline{u}) = 1$.

(iv.) Low Surplus and Low Wage Floor ($\delta < \delta^*$, $\underline{w} < w^*$). The relationship never terminates, and the total payoff of the relationship fluctuates: $\lim_{t\to\infty} u_t + v_t$ *does not exist.*

The long-run outcomes of the relationship follow directly from Proposition 1. Corollary 1 shows that there is a large variety of long-run outcomes. Depending on the wage floor and the surplus level, the relationship may terminate with probability 0, 1, or somewhere in between. In addition, surviving relationship may or may not attain first best in the long run. In contrast, if the principal can commit, the variety of long-run outcomes are more limited. The relationship always survives with positive probability in the long run (because the reward region is absorbing). Moreover, if the relationship survives in the long run, it attains first best. Consequently, the principal's ability to commit affects the dynamics of the relationship, and we discuss its effects in detail in the next section.

4 Implications

The model generates several predictions on turnover and pay dynamics that are consistent with empirical findings. As summarized in the discussion following Proposition 1, our analysis shows that the turnover rate can be inverse-U-shaped, that pay increases with tenure, and that pay is less sensitive to performance in the initial stage of employment. The first two sets of predictions are well documented; see Farber (1994) for a survey on turnover patterns and Rubinstein and Weiss (2006) on wage patterns. There is also some evidence that pay may be more sensitive to performance over time, although the evidence is not exclusively supportive.[8]

These patterns on turnover and pay are also consistent with many other models. The inverse-U-shaped turnover can result from learning models; see, for example, Jovanovic (1979). The upward-sloping wage profile can result from any models of human-capital accumulation. The increasing sensitivity of pay to performance is often explained by career-concern models; see, for example, Gibbons and Murphy (1992) and Gompers and Lerner (1999). In addition, these patterns are consistent with other dynamic moral hazard models with stochastic output; see, for example, Albuquerque and Hopenhayn (2004) and Sannikov (2008). A key feature of our model is that the principal cannot commit. The lack of commitment, combined with the backloading of payments, generates new implications on the dynamics of the relationship, which we discuss next.

To facilitate the discussion on the impact of non-commitment, denote $f_R(\cdot)$ and $f_{LT}(\cdot)$ as the payoff frontier under the optimal relational and long-term contracts, respectively. Using a similar argument as in the proof of Lemma 1, it can be shown that, same as $f_R(\cdot)$, $f_{LT}(\cdot)$ also consists of a punishment-, a probationary-, and a reward region, divided by two cutoffs $u^*_{LT,1}$ and $u^*_{LT,2}$. The punishment region is again linear and can be supported with randomization between $(\underline{u}, f_{LT}(\underline{u}))$ and $\left(u^*_{LT,1}, f_{LT}\left(u^*_{LT,1}\right)\right)$. In addition, there exists a cutoff wage floor $w^*_{LT} < \underline{u}$

[8] Hashimoto (1979) finds that the bonus to wage ratio is increasing with experience in Japanese firms. Gibbons and Murphy (1992) show that the pay of older CEOs is more sensitive to stock-market performance. Gompers and Lerner (1999) document that the sensitivity of pay to performance is smaller for newer venture capitalists. Misra, Coughlan, and Narasimhan (2005) find that the salary to total compensation ratio is decreasing with salesperson seniority. However, Khan and Sherer (1990) find that bonuses are more sensitive to performance for less senior managers.

such that if $\underline{w} \leq w^*_{LT}$, $(\underline{u}, f_{LT}(\underline{u}))$ is supported with suspension, and if $\underline{w} > w^*_{LT}$, $(\underline{u}, f_{LT}(\underline{u}))$ is supported with exit. In the probationary region, the agent puts in effort and is paid $\underline{w}$. In the reward region, the agent puts in effort and receives wage above $\underline{w}$. The reward region is linear with slope -1.

When the relationship has high surplus ($\delta \geq \delta^*$), the principal's inability to commit does not affect the PPE payoff frontier, and therefore, also does not affect the dynamics of the relationship. In particular, $u^*_{LT,1} = u^*_{R,1}$, $u^*_{LT,2} = u^*_{R,2}$, and $w^*_R = w^*_{LT}$, where $u^*_{R,1}$, $u^*_{R,2}$,and w^*_R are the corresponding cutoffs under non-commitment. This follows because relational contracts can be thought of as long-term contracts with the extra non-reneging constraints from the principal. For large enough discount factors, the non-reneging constraints are slack, making the relational contracts identical to the long-term contracts.[9]

When $\delta < \delta^*$, even if the right boundary of the punishment region remains unchanged ($u^*_{LT,1} = u^*_{R,1}$), non-commitment leads to important differences in the PPE payoff frontier, and therefore, the dynamics of the relationship. These differences are summarized in Proposition 2.

PROPOSITION 2: *If* $\delta < \delta^*$, *then* $w^*_R < w^*_{LT}$, $u^*_{R,2} < u^*_{LT,2}$, *and the following holds.*

(*i.*) *Relationships under the optimal relational contracts are less likely to survive in the long run.*

(*ii.*) *Among the surviving relationships, total surplus and expected wages are lower in the optimal relational contract.*

(*iii.*) *The total payoff of the relationship fluctuate in the optimal relational contract, but converges with probability 1 under the optimal long-term contracts.*

Part (i) of the proposition shows that lack of commitment makes the relationship less likely to survive in the long run. There are two reasons for this. First, lack of commitment constrains the size of monetary reward by the amount of future surplus. When $\delta < \delta^*$, the reward alone is not enough to motivate the agent, and the principal must always keep the threat of termination. This implies the agent never receives tenure and a sufficiently long sequence of failures leads to termination. In contrast, when the principal can commit, the size of the reward is no longer constrained. As a result, the reward is sufficiently backloaded that by the first time the principal pays out the reward, the relationship will no longer terminate.

Second, lack of commitment lowers the survival probability by making termination a more likely choice for punishment. Lack of commitment reduces the value of the relationship, making

[9] One caveat is that since there are infinitely many nonreneging constraints–one for each history–it is possible for the optimal long-term and relational contracts to differ for all δ in general; see, for example, Harris and Holmstrom (1982), Thomas and Worrall (1988), and Li and Matouschek (2013). The key condition for the two to be the same is that the principal's continuation payoffs must be strictly and uniformly (for all histories) above her outside option whenever she can renege.

termination a less costly way to punish. When $\underline{w} \in (w_R^*, w_{LT}^*)$, the long-term contract uses suspension to punish and the relational contract uses termination. In this case, the relationship terminates with probability 1 under the relational contract and never terminates under the optimal long-term contract. Figure 3 illustrates the punishment choices under the optimal relational contract and long-term contract.

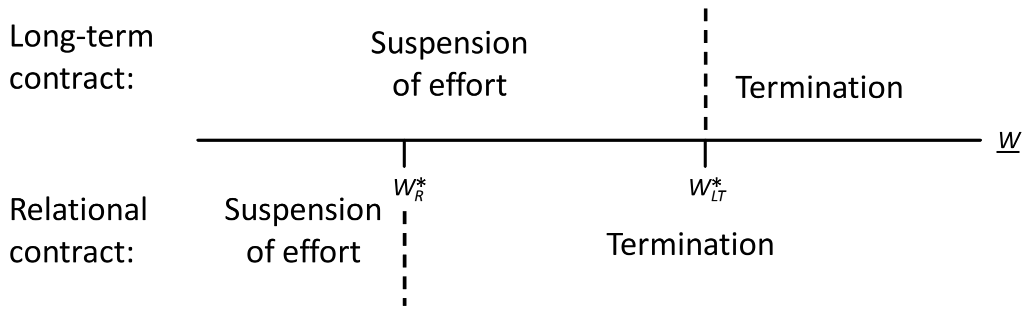

Figure 3: Comparison of Punishment Paths

In addition to making the relationship less likely to survive, lack of commitment also affects the properties of the surviving relationships, as parts (ii) and (iii) of the proposition indicate. Under the optimal long-term contract, the surviving relationships reach the reward region with probability 1. Once the relationship falls into this region, it never leaves, and the relationship is efficient. In contrast, the reward region is non-absorbing under the relational contract when $\delta < \delta^*$. As a result, the agent's continuation payoff will fall into the punishment region following sufficiently many consecutive low outputs, which implies that the agent's payoff cycles between the reward and punishment regions. Consequently, the per-period total payoff of the relationship fluctuates between $py - c$ (when the agent puts in effort) and qy (when suspension is used). In addition, the agent's pay also fluctuates. Note that the agent's expected pay is lower both because he is sometimes suspended and because even if he is asked for effort, the size of the monetary reward is smaller (since it is constrained by the principal's inability to commit).

The long-run difference in the agent's pay is likely to be larger if the moral hazard problem is more severe. To see this, suppose q (the probability of a high output when the agent shirks) increases but the production environment is otherwise unchanged. Under the relational contract (when $\delta < \delta^*$), the highest wage the agent can receive will not change because it is constrained by the surplus of the relationship, which has not changed. Under the long-term contract, however, the wage following a high output is $\underline{w} + (1 - \delta)\, c/(\delta\,(p - q))$, which increases in q.

Proposition 2 suggests that there are empirically distinguishable differences between relationships governed by long-term contracts and those governed by relational contracts. Direct evidence for the theory, however, is difficult to obtain because firms with different levels of commitment power typically differ in other aspects of the production environment, making it difficult to attribute observed differences to the variations in commitment power. One setting in which firms have a similar production environment but differ in their commitment power is the franchise industry, where company-owned outlets might have more commitment power than franchisee-owned ones. In addition, wages are low, and many workers receive minimum wages in this industry. In this case, the minimum wage becomes the relevant limited liability constraint.

Our theory then suggest that company-owned outlets can better backload the compensation of their workforce, resulting in higher wages in the long run, and relatedly, steeper wage-tenure profiles. Two studies examining these differences are Krueger (1991) for fast food restaurants and Freedman and Kosova (2014) for hotels. Krueger (1991) found that the starting pay of lower-level managers is about the same at company-owned and franchisee-owned restaurants, but their pay increases more rapidly at company-owned restaurants. Freedman and Kosova (2014) found that, for an occupation at a hotel, the gap between the highest wage paid and starting wage is higher at company-managed properties.

Our theory also suggests that the long-run differences in pay are likely to be larger when the moral hazard problem is more severe. In this regard, Freedman and Kosova (2014) argued that the activities of housekeepers are more difficult to monitor than those of the front-desk agents. Using the gap between the highest wage paid and starting wage as a measure for the degree of back-loading, they found that the degrees of back-loading (between franchised and company-owned hotels) differ more for housekeepers than for front-desk agents. Relatedly, Krueger (1991) reported that for full-time crew workers, whose activities are easier to monitor than those of lower-level managers, pay dynamics do not differ across company-owned and franchised stores.

Finally, our theory predicts that a lack of commitment reduces the survival probability of the relationship. The evidence on this is more scant. Krueger (1991) found that lower-level managers have a half year longer tenure at company-owned stores. For the crew workers, whose activities are more routine, there is no difference in tenure.[10] Freedman and Kosova (2014) do not have data on worker tenure.

While these findings are consistent with our theory, there are also alternative explanations. Notably, the monitoring intensity is likely to be higher in franchised outlets, since the franchisee, as the owner of the outlet, has a stronger incentive to monitor the workers vis-à-vis the managers in company-owned outlets. The difference in monitoring intensity is clearly relevant, especially

[10] To the extent that the work of the crew is routine, their moral hazard problem is close to our model with p=1. When p=1, our model implies that there is no equilibrium turnover and the relationship is efficient (when it can be sustained). In this case, commitment does not affect employment dynamics. We discussed the case of p=1 in an earlier version of the paper, and the analysis is available upon request.

for fast food restaurants. It is less clear, however, whether the difference in the pay dynamics of hotel workers is entirely due to the difference in monitoring intensity. The hotels in Freedman and Kosova (2014) have 127 rooms on average and include more than 30 types of occupations.[11] As a result, the amount of monitoring each worker received from the franchisee is likely to be quite limited. Of course, the difference in monitoring intensity can still be relevant, as the franchisee may be more motivated to monitor the lower-level managers who in turn monitor the workers. Nevertheless, the difference in monitoring intensity between company-owned and franchised outlets should decrease with the size of the outlets.

5 Extension

In this section, we consider three extensions by allowing the output to be multi-valued, the principal to be more patient than the agent, and the effort level to be continuous. Our main results in the previous section are robust to these extensions. In particular, the payoff frontier has three regions. The agent's wage is equal to the wage floor in the probationary region; his wage is higher in the reward region; and he is terminated or suspended with some probability in the punishment region. The extensions also lead to some differences in the dynamics of the relationship, which we discuss below. The online appendix contains the formal analysis.

5.1 Multi-valued Output

Suppose there are N output levels $y_1 < ... < y_N$. Output y_i occurs with probability p_i if the agent works and with probability q_i otherwise. We assume that MLRP holds so that p_i/q_i is increasing in i, and let m be the cutoff output level such that $p_i/q_i < 1$ for $i \leq m$. Without the limited liability constraint, Levin (2003) shows that the multi-valued output setup is equivalent to a binary one with $p = \sum_{i=m+1}^{N} p_i$ and $q = \sum_{i=m+1}^{N} q_i$.

This equivalence no longer holds when the limited liability constraint is present. In the probationary region, the agent's continuation payoff is no longer binary but is rather increasing in the output level y_i. In the reward region, the agent's continuation payoff again fails to be binary in general. Let u_2^* and $\overline{u}$ be the boundaries of the reward region. When the first best cannot be reached, $u_i\left(u_2^*\right) \leq u_2^*$ for all $i \leq m$ and $u_i\left(u_2^*\right) = \overline{u}$ for all $i \leq m$. The reason that the continuation payoffs are multi-valued is that under the limited liability constraint, the payoff frontier is not a straight line, so the agent's continuation payoffs are typically not bang-bang.

When the first best can be reached, there exists an $n > m$ such that $u_i\left(u_2^*\right) = u_2^*$ for all $i < n$, and $u_i\left(u\right) = \overline{u}$ for all $i > n$. This implies that the relational and long-term contracts can differ even if first best is reached in the long run. Under the optimal long-term contract, it

[11] Freedman and Kosova (2014) do not report the average number of employees per hotel.

can be shown that the agent will be rewarded only when the best output y_N occurs. Rewarding for only the best output allows the principal to minimize the incentive rent left to the agent. Rewarding for only the best output, however, may lead the principal to renege when she cannot commit. The lack of commitment implies that even if the relationship survives in the long run, the rewards are more frequent but are smaller in size under the relational contract.

5.2 More Patient Principal

Suppose the agent's discount factor remains at δ and the principal's discount factor is $\rho > \delta$. For simplicity, we consider the case in which termination will be used as the form of punishment. In this case, one key difference from the main model is that the slope of the reward region changes to $-(1-\rho)/(1-\delta)$. This change affects how the optimal relational contract is implemented. When the principal and the agent are equally patient, there are many ways to implement the reward region since the timing of payment is irrelevant. When the principal is more patient, there is strict gain in paying the agent earlier, so the payment is uniquely determined.

Another difference is that the boundaries separating the three regions are different when the principal is more patient. In the main model, the left boundary of the probationary region is always smaller than its right boundary. When the principal is more patient, in contrast, the probationary region can collapse to a single point. Relatedly, the reward region is self-absorbing when the relationship has a high enough surplus. This may no longer hold when the principal is more patient.

The reason for these differences is that when the principal is more patient than the agent, there is mutual gain in conducting intertemporal trade by paying the less patient agent earlier and letting the more patient principal consume later. As a result, the principal has an incentive to frontload the payment to the agent. Recall that in the main model, the optimal relational contract is driven by the principal's incentive to backload the payment. When the principal is more patient than the agent, the principal must balance the incentive to backload with that to frontload, generating new properties for the optimal relational contract.[12]

First, the optimal relational contract can be efficient, and the optimal relational contract is essentially stationary. This happens when the principal is considerably more patient than the agent. In this case, the incentive to frontload is so important that it completely dominates the incentive to backload. The agent is rewarded as early as possible so that his pay is above the wage floor if and only if the output in the previous period is high. Second, the optimal relational contract can terminate with probability 1 even if there is high surplus. This happens because there is gain in rewarding the agent with monetary payment earlier, so less job security

[12] Opp and Zhu (2015) study the interplay of these two forces in a general environment. They show that the relationship may fluctuate even if the environment has no uncertainty.

is offered. When the agent does not receive tenure, an arbitrarily long sequence of consecutive low outputs will then drive the relationship to the punishment region. Since such long sequences of low outputs happen with probability 1, the relationship will terminate in the long run.

These properties of the optimal relational contract create a new channel for commitment power to affect the relationship. The lack of commitment can lower the survival probability of the relationship by altering the structure of the dynamics. In particular, when the principal is considerably more patient than the agent, the discussion above implies that the optimal long-term contract is efficient and is essentially stationary. Under the relational contract, however, the agent's reward under this type of arrangement can be higher than the surplus of the relationship. In this case, the optimal relational contract becomes inefficient and non-stationary. A low output will lower the agent's continuation payoff, and the relationship terminates with probability 1 in the long run.

5.3 Continuous Effort

We assume that the agent can choose effort $e \in [0, 1]$. When effort e is chosen, the high output is realized with probability $p(e)$, where $p'(e) > 0$ and $p''(e) \leq 0$. In addition, the agent incurs a cost of $c(e)$, where $c(e)$ is increasing and convex with $c(0) = 0$. Just as in the main model, we assume that

$$p(0)y < \underline{u} + \underline{v} < p(e^{FB})y - c(e^{FB}),$$

so the relationship is valuable only when a sufficiently high level of effort is chosen.

The dynamics of the relationship depend on the effort chosen. The online appendix provides a set of necessary conditions for the effort levels. But there is no explicit characterization of the effort level. We can no longer guarantee that randomization is needed in the probationary region. There also appears to be no general rule for the effort dynamics: it is not clear, for example, whether the agent's effort is monotone in his continuation payoff.

Despite the difficulty in determining the effort levels, we show in the online appendix that termination can still occur. Notice that when the agent's payoff goes to his outside option, the level of feasible effort goes to zero. A sufficient condition for termination to occur is that as the agent's effort goes to zero, the probability of the output being high also goes to zero. In this case, termination occurs, because, first, for low enough effort level, the sum of the payoffs through termination exceeds the joint payoff within the relationship for the current period. Second, a low enough effort level implies that the probability of a low output is likely to occur by the condition. As a result, the agent's continuation payoff is likely to drop, and the effort level will continue to be small in the next period. In other words, once the agent's payoff is sufficiently low, the effort levels are likely to continue to be low for a large number of periods if termination is not used. Since this results in a lower payoff than the outside option, it is dominated by termination.

6 Conclusion

This paper studies a model of relational contracts with limited liability. We characterize the optimal relational contract and explore the consequence of non-commitment. The principal's inability to commit makes the relationship less likely to survive, and changes the long-run outcomes of the surviving relationships. Whereas the total surplus of the relationships under optimal long-term contracts always converge to the first best in the long run, it fluctuates under the optimal relational contracts.

One natural extension is to consider multiple firms and workers. Under this extension, the optimal relational contract could become renegotiation-proof.[13] When the firm can costlessly find a replacement in the case of agent termination, the punishment region in the PPE payoff becomes a flat line instead of an upward-sloping one. When the agent's continuation payoff falls into the punishment region, the principal is indifferent between keeping the current agent and finding a new one, and, hence, no renegotiation takes place.

Another possible consequence of the extension is that multiple equilibrium turnover patterns can arise.[14] The reason is that turnover patterns affect the outside options, which in turn affect the surplus in the relationship, which affects the turnover pattern. Consider, for example, an economy in which vacant firms and unemployed workers match randomly. When it becomes easier to form new employment relationships, the surplus in the existing relationship is lowered due to the higher outside options. Our analysis implies that the relationships are more likely to dissolve when the surplus is lower. This increases the number of vacant firms and unemployed workers, making it even easier to form new employment relationships. Such multiplicity may help shed light on the large cross-country differences in employment patterns.

[13] Thomas and Worrall (1994) also considered the relational contracts that are renegotiation-proof. They show that the characterization of the renegotiation-proof contract is unchanged when there is no uncertainty. A corresponding result here is that when $p = 1$, the optimal relational contract is renegotiation-proof along the equilibrium path. When $p < 1$, however, the optimal relational contract is not renegotiatio-proof.

[14] MacLeod and Malcomson (1989, 1998) consider relational contracts with multiple workers and firms when information is public. They show that multiple equilibria arise although the multiplicity does not apply to the turnover rates but rather the different ways to divide the surplus of the relationship.

References

[1] Abreu, Dilip, David Pearce, and Ennio Stacchetti (1990), "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," Econometrica, 58(5), pp. 1041-1063.

[2] Akerlof, George and Lawrence Katz (1989), "Workers' Trust Funds and the Logic of Wage Profiles," Quarterly Journal of Economics, 104 (3), pp. 525-536.

[3] Albuquerque, Rui, and Hugo A. Hopenhayn (2004), "Optimal Lending Contracts and Firm Dynamics," Review of Economic Studies, 71(2), pp. 285-315.

[4] Athey, Susan and Kyle Bagwell (2001), "Optimal Collusion with Private Information," RAND Journal of Economics, 32 (3), pp. 428-465.

[5] Biais, Bruno, Thomas Mariotti, Guillaume Plantin, and Jean-Charles Rochet (2004), "Dynamic Security Design," CEPR Discussion Paper No 4753.

[6] Biais, Bruno, Thomas Mariotti, Guillaume Plantin, and Jean-Charles Rochet (2007), "Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications," Review of Economic Studies, 74(2), pp. 345-390.

[7] Biais, Bruno, Thomas Mariotti, and Jean-Charles Rochet (2013), "Dynamic Financial Contracting": In Daron Acemoglu, Manuel Arellano, and Eddie Dekel (Ed.), *Advances in Economics and Econometrics; Tenth World Congress, Volume 1: Economic Theory*, Cambridge, UK: Cambridge UP, 125-171.

[8] Bull, Clive (1987), "The Existence of Self-Enforcing Implicit Contracts", Quarterly Journal of Economics, 102 (1), pp. 147-159.

[9] Clementi, Gina, and Hugo A Hopenhayn, (2006), "A Theory of Financing Constraints and Firm Dynamics," Quarterly Journal of Economics, 121(1), pp. 229-265.

[10] DeMarzo, Peter and Mike Fishman (2007), "Optimal Long-Term Financial Contracting with Privately Observed Cash Flows," Review of Financial Studies, 20 (6), pp. 2079-2128.

[11] DeMarzo, Peter and Yuliy Sannikov (2006), "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," Journal of Finance, 61, pp. 2681–2724.

[12] Farber, Henry (1994), "The Analysis of Interfirm Worker Mobility," Journal of Labor Economics, 12 (4), pp. 2681–2724.

[13] Freedman, Matthew and Renáta Kosová (2014), "Agency and Compensation: Evidence from the Hotel Industry," Journal of Law, Economics, and Organization, 30(1), pp. 72-103.

[14] Fuchs, William (2007), "Contracting with Repeated Moral Hazard and Private Evaluations," American Economic Review, 97(4), pp. 1432-1448.

[15] Fudenberg, Drew and David K. Levine (1994), "Efficiency and Observability with Long-Run and Short-Run Players," Journal of Economic Theory, 62, pp. 103-135.

[16] Gibbons, Robert and Kevin J. Murphy (1992), "Optimal Incentive Contracts in the Presence of Career Concerns: Theory and Evidence," Journal of Political Economy, 100 (3), pp. 468-505.

[17] Gompers, Paul and Josh Lerner (1999), "An Analysis of Compensation in the U.S. Venture Capital Partnership," Journal of Financial Economics, 51(1), pp. 3-44.

[18] Green, Edward and Robert H. Porter (1984), "Noncooperative Collusion Under Imperfect Price Information," Econometrica, 52(1), pp. 87-100.

[19] Halac, Marina (2012), "Relational Contracts and the Value of Relationships," American Economic Review, 102 (2), pp. 750-779.

[20] Harris, Milton and Bengt Holmstrom (1982), "A Theory of Wage Dynamics," Review of Economic Studies, 49 (3), pp. 315-333.

[21] Hashimoto, Masanori (1979), "Bonus Payments, on-the-Job Training, and Lifetime Employment in Japan," Journal of Political Economy, 87 (5), pp. 1086-1104.

[22] Hörner, Johannes and Larry Samuelson (Forthcoming) "Dynamic Moral Hazard without Commitment," International Journal of Game Theory.

[23] Jovanovic, Boyan (1979), "Firm-Specific Capital and Turnover," Journal of Political Economy, 87 (6), pp. 1246-1260.

[24] Kahn, Lawrence and Peter Sherer (1990), "Contingent Pay and Managerial Performance," Industrial and Labor Relations Review, 43 (3), pp. 107S-120S.

[25] Krueger, Alan (1991), "Ownership, Agency, and Wages: An Examination of Franchising in the Fast Food Industry," Quarterly Journal of Economics, 106 (1), pp. 75-101.

[26] Lazear, Edward (1979), "Why Is There Mandatory Retirement?" Journal of Political Economy, 87 (6), pp. 1261–1284.

[27] Levin, Jonathan (2003), "Relational Incentive Contracts," American Economic Review, 93 (3), pp. 835-57.

[28] Li, Jin and Niko Matouschek (2013), "Managing Conflicts in Relational Contracts," American Economic Review, 103 (6), pp. 2328-2351.

[29] MacLeod, Bentley and James Malcomson (1988), "Reputation and Hierarchy in Dynamic Models of Employment," Journal of Political Economy, 96 (4), pp. 832-854.

[30] MacLeod, Bentley and James Malcomson (1989), "Implicit Contracts, Incentive Compatibility, and Involuntary Unemployment," Econometrica, 57 (2), pp. 447-480.

[31] MacLeod, Bentley and James Malcomson (1998), "Motivation and Markets," American Economic Review, 88 (3), pp. 388-411.

[32] Mailath, George and Larry Samuelson (2006), "Repeated Games and Reputations: Long-run Relationships," Oxford University Press, USA, 2006.

[33] Misra, Sanjog, Anne Coughlan, and Chakravarthi Narasimhan (2005), "Salesforce Compensation: An Analytical and Empirical Examination of the Agency Theoretic Approach," Quantitative Marketing and Economics, 3, pp. 5-39.

[34] Opp M. Marcus and John Y. Zhu (2015), "Impatience versus Incentives," Econometrica, 83 (4), pp. 1601-1617.

[35] Padro i Miguel, Gerard and Pierre Yared (2012), "The Political Economy of Indirect Control," Quarterly Journal of Economics, 127, pp. 947-1015.

[36] Ray, Debraj (2002), "The Time Structure of Self-Enforcing Agreements," Econometrica, 70(2), pp. 547-582.

[37] Rubinstein, Yona and Yoram Weiss, "Post Schooling Wage Growth: Investment, Search and Learning," Chapter 1 in Eric Hanushek and Finis Welch, eds. Handbook of the Economics of Education, Vol. 1, North Holland (2006).

[38] Sannikov, Yuliy (2008), "A Continuous-Time Version of the Principal-Agent Problem," Review of Economic Studies, 75(3), pp. 957-984.

[39] Sannikov, Yuliy (2013), "Dynamic Security Design and Corporate Financing," Handbook of Economics and Finance, 2, pp. 71-122.

[40] Shapiro, Carl and Joseph Stiglitz (1984), "Equilibrium Unemployment as a Worker Discipline Device," American Economic Review, 74(3), pp. 433-444.

[41] Thomas, Jonathan and Tim Worrall (1988), "Self-Enforcing Wage Contracts," Review of Economic Studies, 55(4), pp. 541-554.

[42] Thomas, Jonathan and Tim Worrall (1990), "Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-Agent Problem," Journal of Economic Theory, 51(2), pp. 367-390.

[43] Thomas, Jonathan and Tim Worrall (1994), "Foreign Direct Investment and the Risk of Expropriation," Review of Economic Studies, 61(1), pp. 81-108.

[44] Thomas, Jonathan and Tim Worrall (2010), "Dynamic Relational Contracts with Credit Constraints," Mimeo.

[45] Yang, Huanxing (2013), "Nonstationary Relational Contracts with Adverse Selection," International Economic Review, 54 (2), pp. 525-547

[46] Zhu, John Y. (2013), "Optimal Contracts with Shirking," Review of Economic Studies, 80 (2), pp. 812-839

7 Appendix

Proof of Lemma 1: To prove Lemma 1, it is convenient to first establish a property of the payoff frontier.

Property A: $f'_{+}(u) \geq -1$ for all $u \in [\underline{u}, \overline{u})$. In addition, $f'(u) = -1$ if $w(u) > \underline{w}$, where $w(u)$ is the wage associated with $(u, f(u))$.

To prove $f'_{+}(u) \geq -1$, it suffices to show that $f'_{+}(u) \geq -1$ when $(u, f(u))$ is supported with a pure action. First, suppose $(u, f(u))$ is supported by either suspension or effort. In either case, let $w(u)$ be the associated wage. Now consider an alternative strategy profile with the same action and continuation payoffs but the wage is increased to $w(u) + \varepsilon$ for some small $\varepsilon > 0$. The alternative strategy profile gives a payoff pair of $(u + (1-\delta)\varepsilon, f(u) - (1-\delta)\varepsilon)$, and for small enough ε, it satisfies all the constraints in Section 3.1, and is therefore a PPE payoff. It follows that $f(u) - (1-\delta)\varepsilon \leq f(u + (1-\delta)\varepsilon)$, and sending ε to 0, we have that $f'_{+}(u) \geq -1$. Next, suppose $(u, f(u))$ is supported by exit. In this case, the promise-keeping constraint of the agent and the concavity of f implies that there exists $u' \in (u, \overline{u})$ such that $(u', f(u'))$ is supported with either suspension or effort. By the argument above, we have $f'_{+}(u') \geq -1$, and since $u' > u$ and f is concave, this implies that $f'_{+}(u) \geq f'_{+}(u') \geq -1$. This proves that $f'_{+}(u) \geq -1$ for all $u \in [\underline{u}, \overline{u})$.

Next, we show that if $w(u) > \underline{w}$, then $f'(u) = -1$. To see this, notice that if $w(u) > \underline{w}$, then the same argument above can be applied to the case in which $\varepsilon < 0$, and this proves that $f'_{-}(u) \leq -1$. Now since f is concave, we have $f'_{+}(u) \leq f'_{-}(u) \leq -1$. This, together with $f'_{+}(u) \geq -1$, implies that $f'_{+}(u) = f'_{-}(u) = -1$. This proves Property A.

Part (i): We first show that exists u_1^* such that $f(u)$ is linear in $[\underline{u}, u_1^*]$. This follows from the following two steps.

Step 1: If $(u, f(u))$ is supported with exit for some u, then $f(\underline{u}) = \underline{v}$. In addition, there exists $u_x^* \geq u$ such that $f(\cdot)$ is linear in $[\underline{u}, u_x^*]$. To see this, suppose $(u, f(u))$ is supported with exit. The agent's promise-keeping constraint implies $u_x(u) = (u - (1-\delta)\underline{u})/\delta$. For all $z \in [\underline{u}, \overline{u}]$, define $f_x(z) \equiv (1-\delta)\underline{v} + \delta f(u_x(z))$. This is the highest equilibrium payoff for the principal that is supported by exit and gives the agent a payoff of z. Since $(u, f(u))$ is supported with exit, we have $f(u) = f_x(u)$. Also notice that $f_x(\underline{u}) = \underline{v}$ since $u_x(\underline{u}) = \underline{u}$.

Now for $z > \underline{u}$, we have $u_x(z) = (z - (1-\delta)\underline{u})/\delta > z$. Since f is concave, we have $f'_{x-}(z) = f'_{-}(u_x(z)) \leq f'_{+}(z) \leq f'_{-}(z)$. Consequently,

$$f(\underline{u}) = f(u) - \int_{\underline{u}}^{u} f'_{-}(z)\,dz \leq f_x(u) - \int_{\underline{u}}^{u} f'_{x-}(z)\,dz = f_x(\underline{u}),$$

where recall that $f_x(u) = f(u)$.

Since f is the PPE payoff frontier, we must also have $f(\underline{u}) \geq f_x(\underline{u})$. It then follows that $f(\underline{u}) = f_x(\underline{u}) = \underline{v}$.

Next, since $f(\underline{u}) = f_x(\underline{u})$, the argument above then implies that $\int_{\underline{u}}^{u} f'_{-}(z)\,dz = \int_{\underline{u}}^{u} f'_{x-}(z)\,dz$. As a result, $f'_{-}(z) = f'_{x-}(z) = f'_{-}(u_x(z))$ for almost all $z \leq u$. The concavity of f then implies that f is linear in $[\underline{u}, u_x(u)]$. Let u_x^* be the right end of this line segment. This proves Step 1.

Step 2: We now show that if $(u, f(u))$ is supported with suspension for some u, then $w(u) = \underline{w}$ and $\underline{w} < \underline{u}$. In addition, there exists u_s^* such that $f(u)$ is linear in $[\underline{u}, u_s^*]$.

We first show that, if $(u, f(u))$ is supported with suspension, then $w(u) = \underline{w}$. Suppose to the contrary that $w(u) > \underline{w}$. Then $f'_{+}(u) = -1$ by Property A. In addition, we have $f'_{+}(u_s(u)) = -1$. Otherwise, we have $f'_{+}(u_s(u)) < -1$ (since $f'(u_s(u)) \geq -1$ by Property A). But then consider an alternative strategy profile that is supported with suspension and has $\widehat{w} = w(u) - \delta\varepsilon$ and $\widehat{u}_s = u_s(u) + (1-\delta)\varepsilon$. For small enough $\varepsilon > 0$, this strategy profile is a PPE. It gives the agent a payoff of u. And since $f'_{+}(u_s(u)) < -1$, it gives the principal a payoff that exceeds $f(u)$. This contradicts the definition of f (as the PPE payoff frontier).

Given that $f'_{+}(u_s(u)) = -1$ and $f'_{+}(u) = -1$, it follows u and $u_s(u)$ both lie on the same line segment with slope -1, and therefore, $u + f(u) = u_s(u) + f(u_s(u))$. Adding promise-keeping constraints of the principal and the agent, we then have that

$$\begin{aligned} u + f(u) &= (1-\delta)\,qy + \delta\,(u_s(u) + f(u_s(u))) \\ &= (1-\delta)\,qy + \delta\,(u + f(u)). \end{aligned}$$

This implies that $u + f(u) = qy < \underline{u} + \underline{v}$, which is a contradiction. This proves that $w(u) = \underline{w}$.

Given $w(u) = \underline{w}$, the agent's promise-keeping constraint then implies $u_s(u) = (u - (1-\delta)\,\underline{w})/\delta$. Define

$$f_s(u) \equiv (1-\delta)(qy - \underline{w}) + \delta f(u_s(u)),$$

which is the principal's maximum PPE payoff that gives the agent u and that is supported with suspension and $w(u) = \underline{w}$.

Next, we show that, if $f_s(u) = f(u)$ for some u, then $\underline{u} > \underline{w}$. To do this, we first show that if $f_s(u) = f(u)$ for some u, then $u > \underline{w}$. Suppose to the contrary that $u \leq \underline{w}$. Notice that for all $u' \leq \underline{w}$, $u_s(u') \leq u'$ by $\mathrm{PK_A}$. This implies that $f'_{s+}(u') = f'_{+}(u_s(u')) \geq f'_{+}(u')$ by the concavity of f. It follows that

$$f(\underline{w}) \geq f_s(\underline{w}) = f_s(u) + \int_u^{\underline{w}} f'_{s+}(z)dz \geq f(u) + \int_u^{\underline{w}} f'_{+}(z)dz = f(\underline{w}).$$

We then have $f(\underline{w}) = f_s(\underline{w}) = (1-\delta)(qy - \underline{w}) + \delta f(\underline{w}) = qy - \underline{w}$. Since $\underline{u} < u \leq \underline{w}$, this implies that $f(\underline{w}) < qy - \underline{u} < \underline{v}$, which is a contradiction. This proves that $u > \underline{w}$.

We now show that $\underline{u} > \underline{w}$. To see this, notice that $u_s(u') > u'$ for all $u' \in [\underline{w}, u]$, so

$f'_{s+}(u') = f'_+(u_s(u')) \leq f'_+(u')$. Now suppose to the contrary that $\underline{w} \geq \underline{u}$, then

$$f_s(u) = f_s(\underline{w}) + \int_{\underline{w}}^{u} f'_{s+}(z)dz < f(\underline{w}) + \int_{\underline{w}}^{u} f'_+(z)dz = f(u),$$

where the inequality is strict because $f_s(\underline{w}) < f(\underline{w})$. (Otherwise, $f_s(\underline{w}) = f(\underline{w}) = qy - \underline{w} \leq qy - \underline{u} < \underline{v}$, which is a contradiction.) But this contradicts the assumption that $f_s(u) = f(u)$, and therefore, we cannot have $\underline{w} \geq \underline{u}$. This proves $\underline{u} > \underline{w}$.

Given $\underline{u} > \underline{w}$, we have $f'_{s+}(u') = f'_+(u_s(u')) \leq f'_+(u')$ for all $u' \in [\underline{u}, u]$, so

$$f_s(\underline{u}) = f_s(u) - \int_{\underline{u}}^{u} f'_{s+}(z)dz \geq f(u) - \int_{\underline{u}}^{u} f'_+(z)dz = f(\underline{u}).$$

This chain of inequalities corresponds to that in Step 1, and therefore, f is linear in $[\underline{u}, u_s(u)]$. Let u_s^* be the right end of this line segment. This proves Step 2.

We now prove Part (i) using Step 1 and 2. Note $(\underline{u}, f(\underline{u}))$ is an extremal point of the PPE payoff frontier, so it is supported by a pure action. In addition, $(\underline{u}, f(\underline{u}))$ cannot be supported by effort (because otherwise it would violate Condition LLB). Consequently, $(\underline{u}, f(\underline{u}))$ is supported by either exit or suspension. Step 1 and 2 imply that, if $f(\underline{u}) = \underline{v}$, then $(\underline{u}, f(\underline{u}))$ is supported with exit, and if $f(\underline{u}) > \underline{v}$, it is supported withe suspension. Let $u_1^* = u_x^*$ in the former case and let $u_1^* = u_s^*$ in the later case. This proves that f is linear in $[\underline{u}, u_1^*]$. Also note that $f(\underline{u})$ is weakly decreasing in $\underline{w}$, use $f(u, \underline{w})$ to indicate the dependence of f on $\underline{w}$. It follows that if $f(\underline{u}, \underline{w}) > \underline{v}$ then $(\underline{u}, f(\underline{u}))$ is supported with suspension and the agent is paid $\underline{w}$. This proves Part (i).

Part (ii): Note that by part (i), for all $u \geq u_1^*$, $(u, f(u))$ is not supported by either exit or suspension. In addition, by Property A, $f'(u) = -1$ if $w(u) > \underline{w}$. Now define $u_2^* \equiv \inf\{u : w(u) > \underline{w}\}$. It follows that for all $u \in [u_1^*, u_2^*]$, if $(u, f(u))$ is supported by a pure action, it is supported by effort and $w(u) = \underline{w}$. Next, suppose $(u, f(u))$ is supported by randomization so that $(u, f(u)) = \rho(\widetilde{u}_1, f(\widetilde{u}_1)) + (1-\rho)(\widetilde{u}_2, f(\widetilde{u}_2))$ for some $\rho \in (0,1)$ and $(\widetilde{u}_i, f(\widetilde{u}_i))$, where $(\widetilde{u}_i, f(\widetilde{u}_i))$, $i = 1, 2$, are both supported with pure actions. Because f is concave, we can assume with loss of generality that $u_1^* \leq \widetilde{u}_1 < \widetilde{u}_2 \leq u_2^*$. Consequently, both $(\widetilde{u}_1, f(\widetilde{u}_1))$ and $(\widetilde{u}_2, f(\widetilde{\pi}_2))$ are supported with effort and $w(\widetilde{u}_i) = \underline{w}$, $i = 1, 2$. Now suppose $(\widetilde{u}_i, f(\widetilde{u}_i))$, $i = 1, 2$ are associated with continuation payoffs $(u_{l_i}, f(u_{l_i}))$ and $(u_{h_i}, f(u_{h_i}))$. Consider an alternative strategy profile with first-period wage $\underline{w}$ and continuation payoffs $(\widehat{u}_l, \widehat{v}_l)$ and $(\widehat{u}_h, \widehat{v}_h)$, where $\widehat{u}_l = \rho u_{l_1} + (1-\rho)u_{l_2}$, $\widehat{v}_l = \rho f(u_{l_1}) + (1-\rho) f(u_{l_2})$, and define $\widehat{u}_h$ and $\widehat{v}_h$ analogously. It follows from the promise keeping constraints $\mathrm{PK_P}$ and $\mathrm{PK_A}$ that under this alternative strategy profile the payoffs are given by $\widehat{u} = \rho\widetilde{u}_1 + (1-\rho)\widetilde{u}_2 = u$ and $\widehat{v} = \rho(\widetilde{u}_1, f(\widetilde{u}_1)) + (1-\rho)(\widetilde{u}_2, f(\widetilde{u}_2)) = f(u)$. Moreover, it can be checked that the new strategy profile satisfies all of the constraints in Section 3.1, so it is a PPE. This implies that $(u, f(u))$ can be supported with effort and $w(u) = \underline{w}$. This proves part (ii).

Part (iii): For any $u \geq u_2^*$, Property A implies that $f(u) = f(u_2^*) + u_2^* - u$. Now let $u_{2,l}^*$ and $u_{2,h}^*$

be the agent's continuation payoffs associated with $(u_2^*, f(u_2^*))$. Now for all $u \geq u_2^*$, consider a strategy profile that specifies effort, with $w(u) = \underline{w} + (u - u_2^*)/(1-\delta)$ and continuation payoffs $\left(u_{2,i}^*, f\left(u_{2,i}^*\right)\right)$, $i = l, h$. It can be checked that this strategy profile satisfies all of the constraints in Section 3.1, so it is a PPE. In addition, it gives the agent a payoff of u and the principal a payoff of $f(u)$. Consequently, the payoff frontier can be supported by this strategy profile. This proves part (iii).

Part (iv): Note that $L(u_1^*) + k = u_1^*$ when $\underline{w} = \underline{u} + (1-\delta q) c/(p-q)$. In addition, the self-enforcing condition of the agent implies that $L(u_1^*) \geq \underline{u}$. It then suffices to show that either $L(u_1^*) = \underline{u}$ or $L(u_1^*) + k = u_1^*$. Now suppose $L(u_1^*) > \underline{u}$ and $L(u_1^*) + k \neq u_1^*$. If $L(u_1^*) + k = u_h(u_1^*) < u_1^*$, then $u_l(u_1^*) < u_h(u_1^*) < u_1^*$. Let s be the slope of the payoff frontier in the punishment region. Consider a strategy profile with $\widehat{w} = w(u_1^*)$, $\widehat{u}_l = u_l(u_1^*) + \varepsilon$, $\widehat{u}_h = u_h(u_1^*) + \varepsilon$. For small enough $\varepsilon > 0$, this strategy profile is a PPE and generates a payoff of $(u_1^* + \delta\varepsilon, f(u_1^*) + s\delta\varepsilon)$, contradicting that u_1^* is the right end of the line segment. Now if $L(u_1^*) + k > u_1^*$, then $f'_-(u_h(u_1^*)) < s$ by the definition of u_1^*. Consider a strategy profile with $\widehat{w} = w(u_1^*)$, $\widehat{u}_l = u_l(u_1^*) - \varepsilon$, $\widehat{u}_h = u_h(u_1^*) - \varepsilon$. For small enough $\varepsilon > 0$, this strategy profile is a PPE (because $L(u_1^*) = u_l(u_1^*) > \underline{u}$) and gives the agent an payoff of $\widehat{u} = u_1^* - \delta\varepsilon$. The principal's payoff is given by

$$\begin{aligned}
\widehat{v} &= (1-\delta)(py - \widehat{w}) + \delta((1-p) f(\widehat{u}_l) + p f(\widehat{u}_h)) \\
&= f(u_1^*) + \delta((1-p)(f(\widehat{u}_l) - f(u_l)) + p(f(\widehat{u}_h) - f(u_h))) \\
&> f(u_1^*) - s\delta\varepsilon,
\end{aligned}$$

contradicting that f is the payoff frontier. This proves that either $L(u_1^*) = \underline{u}$ or $L(u_1^*) + k = u_1^*$. This proves part (iv).

Part (v): To determine u_2^*, there are two possibilities. First, when $\delta \geq \delta^*$, it can be checked that the compensation scheme in the proposition leads to a PPE and reaches first best. It then follows that $u_2^* \leq \underline{w} + c/[\delta(p-q)]$, $f(u) = py - c - u$ for $u \geq u_2^*$. In addition, we can assume that IC_A binds by the concavity of f. PK_A then implies that $u_l(u) = L(u)$ and $u_h(u) = L(u) + k$.

Now suppose to the contrary that $u_2^* < \underline{w} + c/[\delta(p-q)]$. It follows that $L(u_2^*) < u_2^*$, and therefore, $L(u_2^*) + f(L(u_2^*)) < py - c$ since u_2^* is the smallest u that reaches first best. But this then implies that $u_2^* + f(u_2^*) < py - c$, a contradiction. Therefore, $u_2^* = L(u_2^*) = \underline{w} + qc/(p-q)$ when $\delta \geq \delta^*$.

Second, when $\delta < \delta^*$, it is clear that $u_2^* < \underline{w} + qc/(p-q)$ (because otherwise we reach first best by the above, contradicting $\delta < \delta^*$). It then follows that $L(u_2^*) < u_2^*$. Recall that $(u_2^*, f(u_2^*))$ is supported with effort with first-period wage $\underline{w}$. Denote its associated continuation payoff as $(u_l, u_h, f(u_l), f(u_h))$, where we may assume $u_l = L(u_2^*)$ and $u_h = L(u_2^*) + k$. Now suppose to the contrary that $u_h < \overline{u}$. Now consider an alternative strategy profile with the same first-period wage $\underline{w}$ but in which the continuation payoffs are given by $(\widehat{u}_l, \widehat{u}_h, f(\widehat{u}_l), f(\widehat{u}_h))$, where

$\widehat{u}_l = u_l + \varepsilon$ and $\widehat{u}_h = u_h + \varepsilon$ for $\varepsilon > 0$. PK_A then implies that the agent's payoff is $\widehat{u} = u_2^* + \delta\varepsilon$. In addition, PK_P implies that under this strategy profile the principal's payoff is given by

$$\begin{aligned} \widehat{v} &= (1-\delta)(py - \underline{w}) + \delta((1-p) f(\widehat{u}_l) + pf(\widehat{u}_h)) \\ &= f(u_2^*) + \delta((1-p)(f(\widehat{u}_l) - f(u_l)) + p(f(\widehat{u}_h) - f(u_h))) \\ &> f(u_2^*) - \delta\varepsilon, \end{aligned}$$

where the strict inequality follows because $f(\widehat{u}_l) - f(u_l) > -\varepsilon$ since $u_l < u_2^*$. Note that the only constraint that new strategy profile tightens is SE_h. But since $u_h < \overline{u}$, SE_h is satisfied for small enough ε, making the new strategy profile a PPE. But this implies $\widehat{v} > f(u_2^*) - \delta\varepsilon = f(\widehat{u})$, contradicting the definition of f. This shows that $L(u_2^*) + k = \overline{u}$ for $\delta < \delta^*$. This proves part (v).■

Proof of Proposition 1: By Lemma 1, $f'_+(u) > 0$ for $u < u_1^*$ and $f'_-(u) < 0$ for $u > u_2^*$, so the payoff frontier is maximized in $[u_1^*, u_2^*]$, and therefore, the agent's first period payoff $u \in [u_1^*, u_2^*]$. In addition, the concavity of f implies that IC_A can be made binding, and the expressions of u_l and u_h follow directly from PK_A. The dynamics of the optimal relational contract then follows readily from Lemma 1.

Finally, we show that for all $u > u_2^*$, we must have $u_l = L(u_2^*)$ and $u_h = L(u_2^*) + k = \overline{u}$ when $\delta < \delta^*$. Suppose this is not the case, IC_Athen implies $u_l < L(u_2^*)$ since $u_h \leq L(u_2^*) + k = \overline{u}$. Recall also that $f'_+(L(u_2^*)) > -1$. Now if IC_A is binding, we have $u_h = u_l + k < \overline{u}$. Consider an alternative strategy profile with the same first-period wage w but in which the continuation payoffs are given by $(\widehat{u}_l, \widehat{u}_h, f(\widehat{u}_l), f(\widehat{u}_h))$, where $\widehat{u}_l = u_l + \varepsilon$ and $\widehat{u}_h = u_h + \varepsilon$ for $\varepsilon > 0$. Sending ε to 0, we then have, by the definition of f,

$$f'_+(u_2^*) \geq (1-p) f'_+(L(u_2^*)) + p(-1) > -1,$$

which is a contradiction.

Next, if IC_A is slack, we have $u_h > u_l + k$. Consider an alternative strategy profile with the same first-period wage w and u_h, but in which $\widehat{u}_l = f(\widehat{u}_l)$. Sending ε to 0, we then have

$$f'_+(u_2^*) \geq f'_+(L(u_2^*)) > -1,$$

which is again a contradiction. This shows that we must have $u_l = L(u_2^*)$ and $u_h = L(u_2^*) + k = \overline{u}$, and PK_A then implies that the associated wage w is also unique. This completes the proof.■

Proof of Corollary 1: The long-run outcomes in the corollary follow directly from Proposition 1 with two caveats. First, we need to rule out that $u_1 = u_2^*$ when $\delta \geq \delta^*$, which would imply that the relationship is efficient and $u_t \geq u_2^*$ for all t. Second, we need to rule out that $u_1 = u_1^*$ when $u_1^* = L(u_1^*) + k$, which would imply that $u_t \leq u_1^*$ for all t. To rule out the first case, note that if $u_1 = u_2^*$ when $\delta \geq \delta^*$, $u_l(u_2^*) = u_2^*$ and $u_h(u_2^*) = u_2^* + k$. Consider a strategy profile with $\widehat{w} = w(u_2^*)$, $\widehat{u}_l = u_l(u_2^*) - \varepsilon$, $\widehat{u}_h = u_h(u_2^*) - \varepsilon$. This gives the agent payoff $\widehat{u} = u_2^* - \delta\varepsilon$ and the principal a payoff

$$\widehat{v} = (1-\delta)(py - w(u_2^*)) + \delta[pf(u_2^* + k - \varepsilon) + (1-p)f(u_2^* - \varepsilon)].$$

Moreover, for small enough $\varepsilon > 0$, this strategy profile is a PPE, so

$$\widehat{v} = f(u_2^*) + \delta[p\varepsilon + (1-p)(f(u_2^* - \varepsilon) - f(u_2^*))] \leq f(\widehat{u}) = f(u_2^* - \delta\varepsilon).$$

Sending ε to 0, the inequality above implies that

$$f'_-(u_2^*) \leq -1.$$

Since $f'_+(u_2^*) = -1$ and f is concave, this implies that $f'(u_2^*) = -1$, and as a consequence, there exists $u < u_2^*$ such that $f(u) > f(u_2^*)$. In other words, $u_1 < u_2^*$.

To rule out the second case, let s be the slope of f between $\underline{u}$ and u_1^*. Note that if $u_1 = u_1^*$ when $u_1^* = L(u_1^*) + k$, we have $u_l(u_2^*) = u_1^* - k$ and $u_h(u_2^*) = u_2^*$. A similar argument as above can then show that $f'_+(u_1^*) = s$, and therefore, $u_1 > u_1^*$. This finishes the proof. ■

Proof of Proposition 2: The payoff frontier of f_{LT} can be characterized in a similar way as f_R, and the frontier again has three regions. The key distinction is that $u_2^* = \underline{w} + qc/(p-q)$ for all δ under the optimal long-term contract, and the reward region is absorbing under the long-term contract. Now to see that $w_R^* < w_{LT}^*$, notice that when $\delta < \delta^*$, $f_{LT}(u) > f_R(u)$ for all $u \in (\underline{u}, \overline{u}]$. By the definition of w_{LT}^*, w_R^*, and the fact that $f_{LT}(u_s(\underline{u})) > f_R(u_s(\underline{u}))$,

$$\begin{aligned}(1-\delta)(qy - w_R^*) f_R(u_s(\underline{u})) &= \underline{v} = (1-\delta)(qy - w_{LT}^*) f_{LT}(u_s(\underline{u})) \\ &> (1-\delta)(qy - w_{LT}^*) f_R(u_s(\underline{u})).\end{aligned}$$

It follows that $w_R^* < w_{LT}^*$.

Now when $\underline{w} > w_{LT}^* (> w_R^*)$, under both the optimal relational contract and long-term contract, the relationship is terminated with a positive probability when u falls below u_1^*. Under the optimal relational contract, a sufficiently long sequence of low outputs (which happens with probability 1 in the long run) will lead u to fall below u_1^*, and, therefore, the relationship terminates with probability 1. Under the optimal long-term contract, there is positive probability that u falls into the reward region ($u \geq u_2^*$), after which the relationship is efficient. When $\underline{w} \in [w_R^*, w_{LT}^*]$, following the same logic as above, the optimal relational contract terminates with probability 1. In contrast, since the optimal long-term contract is punished with suspension, it never terminates. In addition, a sufficiently long sequence of high outputs (which happens with probability 1 in the long run) will lead u to fall above u_2^*, after which the relationship is efficient.

When $\underline{w} < w_R^*$, suspension is used for both the optimal long-term and relational contract. In this case, again the same logic as above shows that the relationship under the optimal long-term contract becomes efficient (when $u \geq u_2^*$) with probability 1. Under the relational contract, the dynamics in Proposition 1 makes it clear that $\liminf_t (u_t) = \underline{u}$ and $\limsup_t (u_t) = \overline{u}$. Part (i) to (iii) follow from the above directly.■

8 Appendix A (For Online Publication)

8.1 Multi-valued Output

Suppose there are N output levels $y_1 < ... < y_N$. Output y_i occurs with probability p_i if the agent works and with probability q_i if the agent shirks. We assume that MLRP holds so that p_i/q_i is increasing in i. Let m be the cutoff output level such that that $p_i/q_i < 1$ for $i \leq m$ and $p_i/q_i > 1$ for $i > m$. Also assume that effort is efficient so that

$$\sum_{i=1}^{N} p_i y_i - c > \underline{v} + \underline{u} > \sum_{i=1}^{N} q_i y_i.$$

As in the main model, we need to specify, for each u,the action of the agent. And if $(u, f(u))$ calls for effort, let $u_i(u)$ as the agent's continuation value if the output is y_i. PK_A then gives

$$u = w(u) - c + \delta \sum_{i=1}^{N} p_i u_i.$$

IC_A is given by

$$\sum_{i=1}^{N} p_i u_i - \sum_{i=1}^{N} q_i u_i \geq c/\delta.$$

Let $p^* = \sum_{m+1}^{N} p_i$, and $q^* = \sum_{m+1}^{N} q_i$. Define

$$\delta^* = \frac{1}{1 + ((p^* - q^*)(p^* y - \underline{w} - \underline{v})/c^* - p^*)}.$$

And similarly, define

$$L(u) = \frac{u}{\delta} - \frac{1-\delta}{\delta}(\underline{w} + \frac{q^*}{p^* - q^*}c).$$

PROPOSITION A0: *There exists u_1^*, and u_2^* that the following holds.*

(i.) For $u \in [\underline{u}, u_1^*)$, $f(u)$ is linear. When $\underline{w} < w^*$ for some $w^* < \underline{u}$, $(\underline{u}, f(\underline{u}))$ is supported with suspension. $w(\underline{u}) = \underline{w}$ and $u_s(\underline{u}) = (\underline{u} - (1-\delta)\underline{w})/\delta$. When $\underline{w} \geq w^*$, $(\underline{u}, f(\underline{u}))$ is supported with exit. When $L(\underline{w} - (1-q)c/(p-q)) \geq \underline{u}$, there exists $n > m$ such that $u_i(u_1^*) = \underline{u}$ for all $i < n$, and $u_i(u_1^*) = u_1^*$ for all $i > n$. *Otherwise, there exists an $n' > m$ such that $u_i(u_1^*) = \underline{u}$ for all $i < n'$, $u_i(u_1^*) \geq u_1^*$ for all $i \geq n'$, and* $\max\{u_i(u_1^*)\} > u_1^*$.

(ii.) For $u \in [u_1^*, u_2^*]$, $(u, f(u))$ can be supported with effort with $w(u) = \underline{w}$. The continuation payoffs satisfy $u_1 \leq ... \leq u$ and

$$f'_-(u_m(u)) \geq f'_+(u).$$

In addition, for $u \in (u_1^*, u_2^*)$, $\min\{u_i(u)\} < u < \max\{u_i(u)\}$.

(iii.) For $u \in (u_2^*, \overline{u}]$, $f(u)$ is linear. $(u, f(u))$ can be supported with effort. $w(u) = \underline{w} + (u -$

$u_2)/(1-\delta)$ and $u_i(u) = u_i(u_2^*)$. When $\delta \geq \delta^*$, first best is reached. There exists an $n > m$ such that $u_i(u_2^*) = u_2^*$ for all $i < n$, and $u_i(u) = \overline{u}$ for all $i > n$. When $\delta < \delta^*$, $u_i(u_2^*) \leq u_2^*$ for all $i \leq m$ and $u_i(u_2^*) = \overline{u}$ for all $i > m$, and the wage payment is unique.

Proof of Proposition A0: For expositional convenience, we start with part (iii.), and then move to part (ii.) and part (i.).
Part (iii.) We first note that $f'_+(u) \geq -1$, and $f'_+(u) = -1$ when $w(u) > \underline{w}$. This follows as the same steps as those in Proposition 1. Now define u_2^* as the left end of the line segment. It follows that $f(u) = f(u_2^*) + u_2^* - u$ for all $u \geq u_2^*$, and $w(u) = \underline{w}$ for all $u \in [u_1^*, u_2^*]$.

To determine u_2^*, there are two possibilities. First, $(u_2^*, f(u_2^*))$ reaches first best. In this case, we must have $u_i(u_2^*) \geq u_2^*$ for all i. Since u_2^* is the left end of the line segment, it is the solution to the following constrained minimization problem:

$$\min_{u_i, u} u$$

such that

$$u = \underline{w} - c + \delta \sum_{i=1}^{N} p_i u_i$$

$$\sum_{i=1}^{N} p_i u_i - \sum_{i=1}^{N} q_i u_i \geq c/\delta.$$

and for all i,

$$py - c - \underline{v} = \overline{u} \geq u_i \geq u.$$

The associated Lagrangian for the constrained minimization problem is given by

$$\begin{aligned} L &= -u + \lambda \left(u - \underline{w} + c - \delta \sum_{i=1}^{N} p_i u_i \right) + \eta \left(\sum_{i=1}^{N} p_i u_i - \sum_{i=1}^{N} q_i u_i - c/\delta \right) \\ &\quad + \sum_{i=1}^{N} \gamma_{i+} (u_i - u) + \sum_{i=1}^{N} \gamma_{i-} (py - c - \underline{v} - u_i) \end{aligned}$$

The FOC with respect of u_i gives that

$$\gamma_{i+} - \gamma_{i-} = \delta p_i \lambda - (p_i - q_i)\eta.$$

It follows that $\gamma_{i+} - \gamma_{i-} > 0$ if and only if $\delta\lambda/\eta > (p_i - q_i)/p_i$. MLRP and complementarity slackness conditions (on γ_{i+} and γ_{i-}) then imply that there exists some $n > m$ such that $u_i(u) = u$ for all $i < n$, and $u_i(u) = \overline{u}$ for all $i > n$. Finally, since IC_A must bind (as in the main model), u and u_n can be solved by $\mathrm{IC_A}$ and $\mathrm{PK_A}$:

$$\begin{aligned}\underline{w} - c + \delta\left(P_{n-1}u + p_n u_n + (1 - P_{n+1})\,\overline{u}\right) &= u; \\ (P_{n-1} - Q_{n-1})\,u + (p_n - q_n)\,u_n + (Q_{n-1} - P_{n+1})\,\overline{u} &= c/\delta,\end{aligned}$$

where $P_n = \sum_{i=1}^{n} p_n$ and $Q_n = \sum_{i=1}^{n} q_n$. This determines u_2^* when first best can be reached.

Next, suppose $(u_2^*, f(u_2^*))$ cannot reach first best. Now suppose to the contrary that $u_i(u_2^*) > u_2^*$ for some $i < m$. Consider an alternative strategy profile with the same wage and continuation value as $(u, f(u))$ except that $\widehat{u}_i = u_i(u) - \varepsilon$ for small $\varepsilon > 0$. This change relaxes $\mathrm{IC_A}$ (since $p_i < q_i$), so the strategy profile is a PPE. It gives the agent again a payoff of $u - \delta p_i \varepsilon$, and the principal a payoff of $f(u) + \delta p_i \varepsilon$ (since $f'(u) = -1$ for all $u \in (u_2^*, \overline{u})$). This contradicts that u_2^* is the left end of the line segment. This shows that $u_i(u_2^*) \leq u_2^*$ for all $i < m$. Moreover, since first best cannot be reached, we may assume WLOG that $u_1(u_2^*) < u_2^*$. This implies that $f'(u_1(u_2^*)) > -1$. Now suppose to the contrary that $u_i(u_2^*) < \overline{u}$ for some $i > m$. Note that $p_i - q_i > 0$. Consider a strategy profile with the same wage and continuation payoffs except that $\widehat{u}_1 = u_1(u_2^*) + (p_i - q_i)/(q_1 - p_1)\,\varepsilon$ and $\widehat{u}_j = u_j(u_2^*) + \varepsilon$. Note that this change preserves $\mathrm{IC_A}$, and therefore for small enough $\varepsilon > 0$ the strategy profile is a PPE. Since $f'(u_1(u_2^*)) > -1$ and $f'(u_i(u_2^*)) \geq -1$, one can check that new PPE lies above the payoff frontier. This is a contradiction. This establishes the property of $u_i(u_2^*)$ when first best cannot be reached.

Finally, we examine the condition for when first best can be reached. To do this, we consider the following dual problem

$$\max_{b_i} \sum_i (p_i - q_i)\, b_i$$

such that for all i,

$$b_i \geq 0$$

and for all i,

$$b_i \leq \frac{\delta}{1-\delta}\left(py - c - \underline{w} - \sum_i p_i b_i\right).$$

It follows that first best can be reached if the solution to this problem is greater than c.

The associated Lagrangian is given by

$$L = \sum_i (p_i - q_i)\, b_i + \sum_{i=1}^{N} \gamma_{i+} b_i + \sum_{i=1}^{N} \gamma_{i-}\left(\frac{\delta}{1-\delta}\left(py - c - \underline{w} - \sum_i p_i b_i\right) - b_i\right).$$

The FOC with respect to b_i then gives that

$$p_i - q_i + \gamma_{i+} - \gamma_{i-}\left(\frac{\delta}{1-\delta} p_i + 1\right) = 0.$$

It then follows that $\gamma_{i+} > 0$ (so $b_i = 0$) when $p_i - q_i < 0$, or $i \leq m$. As a result,

$$b_i = \frac{\delta}{1-\delta}\left(py - c - \underline{w} - \sum_i p_i b_i\right) \equiv b$$

when $i > m$ so that

$$b\left(1 + \frac{\delta}{1-\delta}(1 - P_m)\right) = \frac{\delta}{1-\delta}(py - c - \underline{w}).$$

This implies that the condition for sustaining first is the same as the binary case, with $p = 1 - P_m = p^*$ and $q = 1 - Q_m = q^*$. This finishes the properties of the payoff frontier in part (iii).

Part (ii.) We first show that we can choose $u_1(u) \leq .... \leq u_N(u)$. To see that we can choose $u_i(u) \geq u_j(u)$ for $i > j$, suppose to the contrary that $u_i(u) < u_j(u)$. The concavity of f then implies that $f'_+(u_i(u)) \geq f'_-(u_j(u))$. Consider an alternative strategy profile with the same wage and continuation value as $(u, f(u))$ except $\widehat{u}_i = u_i(u) + p_j\varepsilon$ and $\widehat{u}_j = u_j(u) - p_i\varepsilon$, where $\varepsilon = (u_j(u) - u_i(u))/(p_i + p_j) > 0$ so that $\widehat{u}_i = \widehat{u}_j$. This change relaxes IC_A (since $q_ip_j - p_iq_j < 0$), so the strategy profile is a PPE. It gives the agent again a payoff of u, and since $f'_+(u_i(u)) \geq f'_-(u_j(u))$, it weakly increases the principal's payoff. This implies that $u_n(u)$ can be made weakly increasing in n.

Next, we show that $f'_-(u_m(u)) \geq f'_+(u)$. Suppose to the contrary that $f'_-(u_m(u)) < f'_+(u)$, the concavity of f then implies that $u < u_m$. Consider an alternative strategy profile with the same wage and continuation value as $(u, f(u))$ except that $\widehat{u}_m = u_m(u) - \varepsilon$ for small $\varepsilon > 0$. This change relaxes IC_A (since $p_m < q_m$), so the strategy profile is a PPE. It gives the agent again a payoff of $u - \delta p_m\varepsilon$, and the principal a payoff of $f(u) + \delta p_m(f(u_m(u) - \varepsilon) - f(u_m(u)))$. Since $f'_-(u_m(u)) < f'_+(u)$, the principal's payoff is greater than $f(u - \delta p_m\varepsilon)$, contradicting that f is the PPE payoff frontier. Finally, $\min\{u_i(u)\} < u$ follows from the definition of u_2^*, and $u < \max\{u_i(u)\}$ follows from the definition of u_1^*, which we come to next.

Part (i.) We first determine the condition for the line segment between $(\underline{u}, f(\underline{u}))$ and $(u_1^*, f(u_1^*))$ to be self-sustaining (in the sense that $\max\{u_i(u_1^*)\} \leq u_1^*$). To do this, we examine the dual problem of

$$\max \sum_{i=1}^{N} (p_i - q_i) u_i$$

such that for all i

$$(1-\delta)(\underline{w} - c) + \delta \sum_{n=1}^{N} p_n u_n \geq u_i,$$

and

$$u_i \geq \underline{u}.$$

Let the Lagrangian be

$$L=\sum_{i=1}^{N}(p_i-q_i)\,u_i+\sum_{i=1}^{N}\eta_{i-}\left((1-\delta)\left(\underline{w}-c\right)+\delta\sum_{n=1}^{N}p_nu_n-u_i\right)+\sum_{i=1}^{N}\eta_{i+}\left(u_i-\underline{u}\right).$$

FOC with respect to u_i then gives that

$$p_i-q_i-\eta_{i-}\left(1-\delta p_i\right)+\eta_{i+}=0.$$

It then follows that $\eta_{i+}>0$ (so $u_i=\underline{u}$) when $p_i-q_i<0$, or $i\leq m$. And

$$u_i=(1-\delta)\left(\underline{w}-c\right)+\delta\sum_{n=1}^{N}p_nu_n\equiv u$$

when $i>m$. It follows that the condition for a self-sustaining left region is the same as the binary case with $p^*=\sum_{i=m+1}^{N}p_i$ and $q^*=\sum_{i=m+1}^{N}q_i$.

Next, we determine the value of u_1^* when $(\underline{u},f(\underline{u}))$ and $(u_1^*,f(u_1^*))$ is self-sustaining. In this case, we maximize

$$(1-\delta)\left(\underline{w}-c\right)+\delta\sum_{i=1}^{N}p_iu_i,$$

such that

$$\sum_{i=1}^{N}(p_i-q_i)\,u_i\geq c/\delta$$

and that for all i

$$(1-\delta)\left(\underline{w}-c\right)+\delta\sum_{n=1}^{N}p_nu_n\geq u_i,$$

and

$$u_i\geq\underline{u}.$$

The Lagrangian of the problem is given by

$$\begin{aligned}L\;=\;&(1-\delta)\left(\underline{w}-c\right)+\delta\sum_{i=1}^{N}p_iu_i+\lambda\left(\sum_{i=1}^{N}(p_i-q_i)\,u_i-c/\delta\right)\\&+\sum_{i=1}^{N}\eta_{i-}\left((1-\delta)\left(\underline{w}-c\right)+\delta\sum_{n=1}^{N}p_nu_n-u_i\right)+\sum_{i=1}^{N}\eta_{i+}\left(u_i-\underline{u}\right).\end{aligned}$$

FOC then gives that

$$\delta p_i+\lambda\left(p_i-q_i\right)-\eta_{i-}\left(1-\delta p_i\right)+\eta_{i+}=0.$$

It follows that $\eta_{i+} - (1 - \delta p_i)\,\eta_{i-} > 0$ if $\delta p_i - (p_i - q_i)\,\lambda > 0$, or

$$\frac{\delta}{\lambda} > \frac{(p_i - q_i)}{p_i}.$$

Let

$$u = (1 - \delta)\,(\underline{w} - c) + \delta \sum_{i=1}^{N} p_i u_i.$$

Complementarity slackness then implies that

$$u_i = \begin{cases} \underline{u} & \frac{p_i - q_i}{p_i} < \frac{\delta}{\lambda} \\ & \\ u & \frac{p_i - q_i}{p_i} > \frac{\delta}{\lambda} \end{cases}.$$

In other words, MLRP then implies that there exists an n such that $u_i\,(u) = \underline{u}$ for all $i < n$, and $u_i\,(u) = u$ for all $i > n$. Finally, since $\mathrm{IC_A}$ must bind (as in the main model), u and u_n can be solved by $\mathrm{IC_A}$ and $\mathrm{PK_A}$: u and u_n is then determined by the following equations:

$$\begin{aligned} \underline{w} - c + \delta\,(P_{n-1}\underline{u} + p_n u_n + (1 - P_{n+1})\,u) &= u; \\ (P_{n-1} - Q_{n-1})\,\underline{u} + (p_n - q_n)\,u_n + (Q_{n-1} - P_{n+1})\,u &= c/\delta. \end{aligned}$$

Notice that the analysis above exactly mirrors that in determining u_2^* when $(u_2^*, f\,(u_2^*))$ reaches the first best. Similarly, when $(\underline{u}, f\,(\underline{u}))$ and $(u_1^*, f\,(u_1^*))$ is not self-sustaining (so that $\max\{u_i\,(u_1^*)\} > u_1^*$), similar argument as in part (i) gives that $u_i\,(u_1^*) = \underline{u}$ for $i < n'$ for some $n' > m$ and $u_i\,(u_1^*) \geq u_1^*$ otherwise. This finishes the proof of Part(i). ■

8.2 More Patient Principal

Let the agent's discount factor to remain at δ and let the principal's discount factor be $\rho > \delta$. We assume that conditions LLB and NT (at the end of Section 2) are satisfied, and these conditions guarantee the existence of a nontrivial relational contract in which the limited liability constraint is binding. Notice that these conditions only depend only on δ, the discount factor of the less patient player. We start by describing the PPE payoff frontier.

LEMMA A1: *There exist $u_1^* \leq u_2^*$ such that the PPE frontier $f(u)$ is given by the following.*

(i.) For $u \in [\underline{u}, u_1^*)$,

$$f(u) = \underline{v} + \frac{u - \underline{u}}{u_1^* - \underline{u}}(f(u_1^*) - \underline{v}).$$

(ii.) For $u \in [u_1^*, u_2^*]$,

$$f(u) = (1 - \rho)(py - \underline{w}) + \rho[pf(L(u) + k) + (1 - p)f(L(u))].$$

(iii.) For $u \in (u_2^*, \overline{u}]$,

$$f(u) = f(u_2^*) + \frac{1-\rho}{1-\delta}(u_2^* - u).$$

Proof of Lemma A1: We prove part (i.) first. We show that there exists u_1^* such that f is supported by randomization if and only if $u \in (\underline{u}, u_1^*)$. To see this, again let $u_h(u)$ be the continuation payoff following a high output and $u_l(u)$ the continuation following a failure. To induce effort, the incentive compatibility constraint of the agent requires

$$u_h(u) - u_l(u) \geq k.$$

This implies that if the worker puts in effort, his payoff satisfies

$$\begin{aligned} u &= (1-\delta)(w-c) + \delta(u_l(u) + p(u_h(u) - u_l(u))) \\ &\geq (1-\delta)(\underline{w}-c) + \delta(\underline{u} + pk). \end{aligned}$$

Therefore, for $u \in (\underline{u}, (1-\delta)(\underline{w}-c) + \delta(\underline{u}+pk))$, effort cannot be provided, so the payoff frontier is supported by randomization. It follows that for $u \in [\underline{u}, (1-\delta)(\underline{w}-c) + \delta(\underline{u}+pk)]$, the payoff frontier is a line segment where the left point is $(\underline{u}, \underline{v})$, i.e. $f(u) = \underline{v} + s(u - \underline{u})$ for some $s > 0$. Now denote the right end point of the line segment as $(u_1^*, f(u_1^*))$, i.e.,

$$u_1^* \equiv \max\{u : f(u) = \underline{v} + s(u - \underline{u})\}.$$

By the definition of u_1^*, we have $f(u) = f(\underline{u}) + \frac{u-\underline{u}}{u_1^*-\underline{u}}(f(u_1^*) - \underline{v})$ for all $u \in [\underline{u}, u_1^*]$.

Next, we show that randomization is not needed for all payoff pairs $(u, f(u))$ when $u > u_1^*$. To see this, suppose that

$$(u, f(u)) = \alpha(u_1, f(u_1)) + (1-\alpha)(u_2, f(u_2))$$

where $\alpha \in (0,1)$, and $(u_i, f(u_i))$ are two extremal points on the frontier for $i = 1, 2$ with $u_1 < u_2$. Notice we must have $u_1 > \underline{u}$ because a linear combination using $(\underline{u}, \underline{v})$ and $(u_2, f(u_2))$ is strictly dominated by the corresponding linear combination using $(u_1^*, f(u_1^*))$ and $(u_2, f(u_2))$. Let (w_i, u_{l_i}, u_{h_i}) be the associated wages and continuation payoffs for $i = 1, 2$. Now consider a payoff pair supported by the following. Specifically, the principal pays out $w = \alpha w_1 + (1-\alpha) w_2$ in the first period (and the worker puts in effort), and the continuation payoffs following a high and low output be $(u_l, f(u_l))$ and $(u_h, f(u_h))$, where $u_l = \alpha u_{l_1} + (1-\alpha) u_{l_2}$ and $u_h = \alpha u_{h_1} + (1-\alpha) u_{h_2}$. It is clear that this payoff pair is a PPE payoff and it gives the agent a payoff of u. Moreover, it gives the principal a payoff of

$$\begin{aligned}
& (1-\rho)(py-w)+\rho\left(pf(u_h)+(1-p)f(u_l)\right) \\
\geq \;& \alpha f(u_1)+(1-\alpha)f(u_2)
\end{aligned}$$

by the concavity of f. This shows that the payoff frontier can be supported by pure strategies for all $u \geq u_1^*$.

We prove part (iii.) next. To see this, we first show that $f'_+(u) \geq -(1-\rho)/(1-\delta)$ for all u. Notice that for $u < u_1^*$, this is clearly satisfied since $f'(u) > 0$ for $u < u_1^*$ by (i). For $u \in [u_1^*, \overline{u}]$, part (i.) above shows that $(u, f(u))$ can be supported by pure actions. Moreover, it is clear that $f(u) > \underline{v}$. Now suppose $(u, f(u))$ is supported by a wage w in the first period and the continuation payoffs $(u_i, f(u_i))$ for $i = l, h$. Consider a new payoff pair supported by first-period wage $w+\varepsilon$ and the same continuation payoffs. For small enough positive ε, this new set of actions and continuation payoffs satisfy all of the constraints and constitutes a PPE. Moreover, this PPE payoff is given by $(u+(1-\delta)\varepsilon, f(u)-(1-\rho)\varepsilon)$. By the definition of f, we have

$$f(u+(1-\delta)\varepsilon) \geq f(u)-(1-\rho)\varepsilon.$$

Sending ε to 0, we obtain $f'_+(u) \geq -(1-\rho)/(1-\delta)$. Now define $u_2^* = \inf\{u, f'(u) = -\frac{1-\rho}{1-\delta}\}$. This proves (iii).

Finally, we prove part (ii.). In particular, suppose $(u, f(u))$ is supported by (w, u_l, u_h), we need to show that $w = \underline{w}$, $u_l = L(u)$ and $u_h = L(u)+k$ for all $u \leq u_2^*$. We first show that $w(u) = \underline{w}$. To see this, suppose to the contrary that $w > \underline{w}$. Then the same argument as in the proof of part (iii.) shows that $(u+(1-\delta)\varepsilon, f(u)-(1-\rho)\varepsilon)$ is again a PPE payoff for small negative ε, and therefore, $f'_-(u) \leq -(1-\rho)/(1-\delta)$. Since $f'_-(u) \geq f'_+(u)$, we must have $f'(u) = -(1-\rho)/(1-\delta)$. This contradicts the definition of u_2^*, and therefore, we must have $w(u) = \underline{w}$ for all $u \leq u_2^*$. Next, we show $u_l = L(u)$ and $u_h = L(u)+k$. Suppose the contrary. Then the promise-keeping constraint for the agent and the incentive compatibility constraint for the agent must imply that $u_l < L(u)$ and $u_h > L(u)+k$. Now consider a new payoff pair supported by $w = \underline{w}$, $u'_l = L(u)$ and $u'_h = L(u)+k$. For small enough $\varepsilon > 0$, this new set of action and continuation satisfies all constraints and supports a PPE payoff. Moreover, the new PPE gives the agent a payoff of u and the principal a payoff of

$$\begin{aligned}
& (1-\rho)(py-\underline{w})+\rho\left((1-p)f(L(u))+pf(L(u)+k)\right) \\
\geq \;& (1-\rho)(py-\underline{w})+\rho\left((1-p)f(u_l)+pf(u_h)\right),
\end{aligned}$$

where the inequality follows because f is concave, $u_l < L(u) < L(u)+k < u_h$, and

$$(1-p)u_l + pu_h = (1-p)L(u) + p(L(u)+k).$$

By the definition of f, we must then have

$$f(u) = (1-\rho)(py - \underline{w}) + \rho((1-p)f(L(u)) + pf(L(u)+k)).$$

This proves part (ii.). ■

Notice that for f to be the PPE payoff frontier, we should also check that

$$f(u) \geq \max_{\substack{w \geq \underline{w}, u_s \in [\underline{u}, \overline{u}] \\ (1-\delta)w + \delta u_s = u}} (1-\rho)(qy - w) + \rho f(u_s), \quad \text{for all } u \in [\underline{u}, \overline{u}]$$

so that it will not be optimal for the agent to shirk. Since $f'_{-} \geq -(1-\rho)/(1-\delta)$, the above condition is equivalent to

$$f(u) \geq (1-\rho)(qy - \underline{w}) + \rho f(\frac{u-(1-\delta)}{\delta}), \quad \text{for all } u \in [\underline{u}, \overline{u}] \text{ such that } \frac{u-(1-\delta)}{\delta} \in [\underline{u}, \overline{u}].$$

While the inequality above cannot be directly mapped into an inequality that involves exogenous conditions only, a sufficient condition for it to hold is that

$$(1-\rho)(qy - \underline{w}) + \rho(py - c - \underline{w}) \leq \underline{v}.$$

In this case, expected payoff from suspending the agent is so costly for the principal that termination is preferred. To simplify the exposition, we define the following condition.

Definition 1 (Condition A)

$$py - \underline{w} - \underline{v} - \frac{\rho}{\delta}\frac{pc}{p-q} \geq \max\{\frac{1-\rho}{\delta}\frac{c}{p-q}, \frac{1-\rho}{1-\delta}\frac{\rho p}{\rho(1-p)-\delta}\left(\underline{w} + \frac{qc}{p-q} - \underline{u}\right)\} \tag{1}$$

Notice that Condition A is a combination of two inequalities. The first inequality (that the left hand side is bigger than $(1-\rho)c/(\delta(p-q))$ corresponds to the high-surplus condition in the main model. It is satisfied when the surplus is sufficiently high. The second inequality is a technical condition that measures the cost of termination, and it essentially gives a lower bound to the slope of the line segment that links $(\underline{u}, \underline{v})$ and $(u_2, f(u_2))$. Condition A directly leads to the following proposition.

PROPOSITION A1: The *optimal relational contract takes one of the two forms.*

(i.) If $\delta < \rho(1-p)$ *and Condition A is satisfied, the optimal relational contract is quasi-stationary. Specifically, the relationship starts at* $u_2^* = \underline{w} + \frac{qc}{p-q}$. *The agent always puts in effort. Along the equilibrium path,* $w(u) = \underline{w} + (u - u_2^*)/(1-\delta)$, $u_l(u) = u_2^*$, and $u_h(u) = u_2^* + k$.

(ii.) If $\delta \geq \rho(1-p)$ or Condition A fails, $L(u_2^*) < u_2^*$, and the relationship terminates with probability 1. The optimal relational contract starts in $[u_1^*, u_2^*]$ and takes the following form:

(a): For $u \in [\underline{u}, u_1^*)$,the relationship is terminated with probability $(u_1^* - u) / (u_1^* - \underline{u})$ and goes to $(u_1^*, f(u_1^*))$ with probability $(u - \underline{u}) / (u_1^* - \underline{u})$.

(b): For $u \in [u_1^*, u_2^*]$, the agent puts in effort, $w(u) = \underline{w}$, $u_l(u) = L(u)$, and $u_h(u) = L(u) + k$.

(c): For $u \in (u_2^*, \overline{u}]$, the agent puts in effort, $\underline{w} + (u - u_2) / (1 - \delta), u_l(u) = L(u_2^*)$, and $u_h(u) = L(u_2^*) + k$.

The next proposition explores the role of commitment by comparing the optimal relational contracts and long-term contracts when the principal is more patient.

PROPOSITION A2: *The following holds.*

(i.) There exists a ρ^* such that the optimal relational contract and the optimal long-term contract are identical if and only if $\rho \geq \rho^*$.

(ii.) If $\delta < \rho(1-p)$ and

$$\frac{1-\rho}{\delta}\frac{c}{p-q} > py - \underline{w} - \underline{v} - p\frac{c}{p-q}\frac{\rho}{\delta} \geq \frac{1-\rho}{1-\delta}\frac{\rho p}{\rho(1-p)-\delta}\left(\underline{w} + \frac{qc}{p-q} - \underline{u}\right),$$

the optimal long-term contract is quasi-stationary and the optimal relational contract terminates with probability 1.

Notice that to show Proposition A2, it suffices to prove the following lemma.

LEMMA A2: *The followings hold.*

(i.) $u_l(u_2^*) \leq u_2^*$.

(ii.) $u_l(u_2^*) = u_2^*$ if and only if $\delta < \rho(1-p)$

$$py - \underline{w} - \underline{v} - \frac{\rho}{\delta}\frac{pc}{p-q} \geq \max\{\frac{1-\rho}{\delta}\frac{c}{p-q}, \frac{1-\rho}{1-\delta}\frac{\rho p}{\rho(1-p)-\delta}\left(\underline{w} + \frac{qc}{p-q} - \underline{u}\right),\}. \text{ (Condition A)}$$

Proof of Lemma A2: Part (i). First note that $w(u_2^*) = \underline{w}$ and $u_h(u_2^*) - u_l(u_2^*) = k$. Therefore, $L(u_2^*) \leq u_2^*$ is equivalent to $u_l(u_2^*) \leq u_2^*$. Suppose to the contrary that $u_l(u_2^*) > u_2^*$. Consider an alternative strategy profile supported by first-period wage $\underline{w}$ and continuation payoffs $u_l(u_2^*) - \varepsilon$ and $u_h(u_2^*) - \varepsilon$. For small enough $\varepsilon > 0$, the first-period action and continuation payoffs satisfy all constraints and support a PPE payoff. Moreover, the PPE gives the agent a payoff of $u_2^* - \rho\varepsilon$

and the principal a payoff of

$$\begin{aligned} & (1-\rho)(py-\underline{w})+\rho\left((1-p) f\left(u_l(u_2^*)-\varepsilon\right)+p f\left(u_h(u_2^*)-\varepsilon\right)\right) \\ = \;& (1-\rho)(py-\underline{w})+\rho\left((1-p) f\left(u_l(u_2^*)\right)+p f\left(u_h(u_2^*)\right)\right)+\frac{1-\rho}{1-\delta}\rho\varepsilon \\ \leq \;& f\left(u_2^*-\rho\varepsilon\right), \end{aligned}$$

where the last inequality follows the definition of f.

But this implies that

$$f(u_2^*)-f\left(u_2^*-\rho\varepsilon\right) \leq -\frac{1-\rho}{1-\delta}\rho\varepsilon,$$

so the slope between $u_2^*-\rho\varepsilon$ and u_2^* is weakly smaller than $-(1-\rho)/(1-\delta)$. By Lemma A1, this implies that the slope between $u_2^*-\rho\varepsilon$ and u_2^* must be equal to $-(1-\rho)/(1-\delta)$. But this contradicts the definition of u_2^*. This proves part (i).

Part (ii.): We first show that if $u_l(u_2^*)=u_2^*$, it must be the case that $\delta<\rho(1-p)$ and Condition A is satisfied. Notice that when $u_l(u_2^*)=u_2^*$, we have $u_h(u_2^*)=u_2^*+k$. As a result, $u_2^*=\underline{w}+qc/(p-q)$ and $f(u_2^*)=py-\underline{w}-pc\rho/((p-q)\delta)$. Condition A is then equivalent to

$$f(u_2^*)-\underline{v} \geq \frac{1-\rho}{1-\delta}\max\{\frac{\rho p}{\rho(1-p)-\delta}\left(u_2^*-\underline{u}\right), k\}.$$

Notice that $f(\overline{u})=\underline{v}$, and $\overline{u} \geq u_h(u_2^*)=u_2^*+k$. It follows that

$$f(u_2^*)-\underline{v}=f(u_2^*)-f(\overline{u})=\frac{1-\rho}{1-\delta}\left(\overline{u}-u_2^*\right) \geq \frac{1-\rho}{1-\delta}k.$$

It then remains to show that $\delta<\rho(1-p)$ and $f(u_2^*)-\underline{v} \geq \frac{1-\rho}{1-\delta}\frac{\rho p}{\rho(1-p)-\delta}\left(u_2^*-\underline{u}\right)$. Consider a payoff supported with the same actions and the continuation payoffs being $u_l=u_2^*-\varepsilon$, $u_h=u_2^*+k-\varepsilon$, $f(u_l)=f(u_2^*-\varepsilon)$, $f(u_h)=f(u_2^*+k-\varepsilon)$. This alternative profile generates payoff of $u_2^*-\delta\varepsilon$ for the agent, and gives the principal a payoff of

$$v=f(u_2^*)-\rho\left((1-p)\left(f(u_2^*-\varepsilon)-f(u_2^*)\right)+p\left(f(u_2^*+k-\varepsilon)-f(u_2^*+k)\right)\right).$$

By the definition of f, we have $v \leq f\left(u_2^*-\delta\varepsilon\right)$, or equivalently,

$$\begin{aligned} & f(u_2^*)-f\left(u_2^*-\delta\varepsilon\right) \\ \geq \;& \rho\left((1-p)\left(f(u_2^*-\varepsilon)-f(u_2^*)+p\left(f(u_2^*+k-\varepsilon)-f(u_2^*+k)\right)\right)\right). \end{aligned}$$

Sending ε to zero and noticing that $f'(u)=\frac{1-\rho}{1-\delta}$ for $u>u_2^*$, we obtain.

$$\left(\delta-\rho(1-p)\right) f_-'(u_2^*) \leq -\rho p\frac{1-\rho}{1-\delta}.$$

Since the right hand side is negative, it is clear that we cannot have $\delta - \rho(1-p) = 0$. Now if $\delta - \rho(1-p) > 0$, this implies that

$$f'_{-}(u_2^*) \leq -\frac{\rho p}{\delta - \rho(1-p)} \frac{1-\rho}{1-\delta} < -\frac{1-\rho}{1-\delta},$$

contradicting the concavity of f. This proves that if $u_l(u_2^*) = u_2^*$, we must have $\delta - \rho(1-p) < 0$.

Notice that when $\delta < \rho(1-p)$, the argument above implies that

$$f'_{-}(u_2^*) \geq \frac{1-\rho}{1-\delta} \frac{\rho p}{\rho(1-p) - \delta}.$$

Moreover, since for all $u \in [\underline{u}, u_2^*]$, the payoff frontier f is weakly dominated by the randomization between $(\underline{u}, \underline{v})$ and $(u_2^*, f(u_2^*))$. It follows that $f'_{-}(u_2^*) \leq (f(u_2^*) - \underline{v}) / (u_2^* - \underline{u})$. Therefore,

$$\frac{f(u_2^*) - \underline{v}}{u_2^* - \underline{u}} \geq \frac{1-\rho}{1-\delta} \frac{\rho p}{\rho(1-p) - \delta},$$

so

$$f(u_2^*) - \underline{v} \geq \frac{1-\rho}{1-\delta} \frac{\rho p}{\rho(1-p) - \delta} (u_2^* - \underline{u}).$$

This proves the "if" part.

Next, we show that if $\delta < \rho(1-p)$ and Condition A is satisfied, then $u_l(u_2^*) = u_2^*$. To see this, define $f(u)$ as follows.

$$f(u) = \begin{cases} \underline{v} + \frac{u - \underline{u}}{u_1^* - \underline{u}} (f(u_2^*) - \underline{v}) & for\ u \in [\underline{u}, u_2^*] \\ f(u_2^*) + \frac{1-\rho}{1-\delta} (u_2^* - u) & for\ u \in [u_2^*, \overline{u}] \end{cases},$$

where $u_2^* = \underline{w} + \frac{qc}{p-q}$, $f(u_2^*) = py - \underline{w} - p\frac{c}{p-q}\frac{\rho}{\delta}$, and $\overline{u} = u_2^* + (1-\delta)(f(u_2^*) - \underline{v}) / (1-\rho)$.

It is clear that under f, we have $u_l(u_2^*) = u_2^*$. We will show below that f is the payoff frontier of the long-term contracts, and therefore, the payoff frontier of the relational contracts, and this finishes the proof.

Notice that the payoff frontier of the long-term contract must satisfy the following functional equation:

$$f(u) = \max\{f_1(u), f_2(u)\},$$

where

$$f_1(u) = \max_{\substack{u_1, u_2, \alpha \in [0,1], \\ \alpha u_1 + (1-\alpha) u_2 = u}} \{\alpha f(u_1) + (1-\alpha) f(u_2)\},$$

and

$$f_2(u) = \max_{w \geq \underline{w}, u_l, u_h} (1-\rho)(py - w) + \rho((1-p) f(u_l) + p f(u_h))$$

such that

$$\begin{aligned}(1-\delta)(w-c)+\delta((1-p)u_l+pu_h) &= u;\\ u_h-u_l &\geq k.\end{aligned}$$

Moreover, define the operator $Tf \equiv \max\{f_1(u), f_2(u)\}$, where T is defined on bounded functions with support in $[\underline{u}, M]$ It can be checked that T is a, monotone, nonexpansive mapping, and, thus, has a unique fixed point. Therefore, it suffices to show that f constructed above satisfies $Tf = f$.

It is easy to check that $f(u) = \max\{f_1(u), f_2(u)\}$ for $u \geq u_2^*$. For $u < u_2^*$, we have $f(u) = f_1(u)$ by construction. Therefore, the proof is complete once we show that $f(u) \geq f_2(u)$ for $u < u_2^*$. While f_2 is cast as the solution to a maximization problem, argument from argument in the proof of Lemma A1 immediately that the maximizers must satisfy $w = \underline{w}$, $u_l = L(u)$ and $u_h = L(u) + k$ for $u \leq u_2^*$. This implies that for $u \leq u_2^*$, we have

$$f_2(u) = (1-\rho)(py-\underline{w}) + \rho((1-p)f(L(u)) + pf(L(u)+k)).$$

Now notice that $f(u_2^*) = f_2(u_2^*)$ by construction. Moreover, define $s \equiv \frac{f(u_2^*)-\underline{v}}{u_2^*-\underline{u}}$ as the slope of f between $\underline{u}$ and u_2^*, then

$$\begin{aligned}f_2'(u) &= \frac{\rho}{\delta}\left((1-p)f'(L(u)) + pf'(L(u)+k)\right)\\ &\geq \frac{\rho}{\delta}\left((1-p)s + p\left(-\frac{1-\rho}{1-\delta}\right)\right)\\ &\geq s,\end{aligned}$$

where the last inequality follows because it is equivalent to $(\rho(1-p)-\delta)s \geq p\rho(1-\rho)/(1-\delta)$. Since $f(u_2^*) = f_2(u_2^*)$ and $f_2'(u) \geq f'(u)$ for all $u \leq u_2^*$, this implies that $f_2(u) \leq f(u)$ for all $u \leq u_2^*$. It follows that the constructed f is the unique solution to $f = Tf$. From the construction of f, it is clear that $u_l(u_2^*) = u_2^*$. Therefore, we have $u_l(u_2^*) = u_2^*$ if and only if $\delta < \rho(1-p)$ and Condition A holds. ■

Proof of Proposition A2: Part (i.) is straightforward and is omitted. For part (ii.), notice that on the one hand, if $\delta < \rho(1-p)$ and

$$py - \underline{w} - \underline{v} - p\frac{c}{p-q}\frac{\rho}{\delta} \geq \frac{1-\rho}{1-\delta}\frac{\rho p}{\rho(1-p)-\delta}\left(\underline{w} + \frac{qc}{p-q} - \underline{u}\right),$$

we can use the same argument as that in the proof of Lemma A2 to show that the payoff frontier under the long-term contract is given by

$$f_{LT}(u) = \begin{cases} \underline{v} + \frac{u-\underline{u}}{u_1^*-\underline{u}}(f_{LT}(u_2^*) - \underline{v}) & for\ u \in [\underline{u}, u_2^*] \\ f_{LT}(u_2^*) + \frac{1-\rho}{1-\delta}(u_2^* - u) & for\ u \in [u_2^*, \overline{u}] \end{cases},$$

where $u_2^* = \underline{w} + \frac{qc}{p-q}$, $f_{LT}(u_2^*) = py - \underline{w} - p\frac{c}{p-q}\frac{\rho}{\delta}$, and $\overline{u} = u_2^* + (1-\delta)\left(f_{LT}(u_2^*) - \underline{v}\right)/(1-\rho)$. It follows that $u_2^* = L(u_2^*)$, and as a result, the optimal long-term contract is stationary. On the other hand, when

$$\frac{1-\rho}{\delta}\frac{c}{p-q} > py - \underline{w} - \underline{v} - p\frac{c}{p-q}\frac{\rho}{\delta},$$

the condition implies that stationary relational contract is impossible. As a result, $L(u_2^*) < u_2^*$ and the relationship terminates with probability 1. ■

8.3 Continuous Effort

Assume that the agent can choose effort $e \in [0,1]$. When effort e is chosen, the high output is realized with probability $p(e)$, where $p'(e) > 0$ and $p''(e) \leq 0$. In addition, the agent incurs a cost of $c(e)$, where $c(e)$ is increasing and convex with and $c(0) = 0$. Just as in the main model, we assume that

$$p(0)y < \underline{u} + \underline{v} < p(e^{FB})y - c(e^{FB}),$$

so the relationship is valuable only when a sufficiently high level of effort is chosen. The next proposition describes the PPE payoff frontier.

PROPOSITION A3: *There exist $u_1^* \leq u_2$ such that the PPE payoff frontier $f(u)$ is given by the following.*

(*i.*) *For $u \in [\underline{u}, u_1^*)$,*

$$f(u) = f(\underline{u}) + \frac{u - \underline{u}}{u_1^* - \underline{u}}(f(u_1^*) - f(\underline{u})).$$

(*ii.*) *For $u \in [u_1^*, u_2]$,*

$$f(u) = (1-\delta)(p(e)y - \underline{w}) + \delta[p(e)f(u_h(u)) + (1-p(e))f(u_l(u))]$$

when $(u, f(u))$ can be supported with pure action. In this case, $e(u) > 0$, $w(u) = \underline{w}$, and when $c'(e)$ exists,

$$\begin{aligned} u_l(u) &= \frac{1}{\delta}\left(u - (1-\delta)(\underline{w} - c(e) + \frac{p(e)c'(e)}{p'(e)})\right), \\ u_h(u) &= u_l(u) + \frac{1-\delta}{\delta}\frac{c'(e)}{p'(e)}. \end{aligned}$$

(*iii.*) *For $u \in (u_2^*, \overline{u}]$,*

$$f(u) = f(u_2^*) + (u_2^* - u).$$

In this case, we can choose $e(u) = e(u_2^*), w = \underline{w} + (u - u_2^*)/(1-\delta)$, $u_l(u) = u_2^*$, *and* $u_h(u) = u_2^* + (1-\delta)\, c'_-(e(u_2^*))/(\delta p'(e(u_2^*)))$.

Proof of Proposition A3: For expositional convenience, we assume that c is differentiable. When $c'(e)$ does not exist, the argument can be adapted through the use of left-derivative. We prove part (i) first. Define u'_0 as the smallest payoff of the agent in which the payoff frontier requires a positive effort, i.e.,

$$u_1^* = \inf\{u : e(u) > 0\}.$$

then the payoff frontier is a straight line between $(\underline{u}, f(\underline{u}))$ and $(u'_0, f(u'_0))$. Define u_1^* as the right end point of the line segment, and then part(i) follows.

Next, we prove part (iii). Notice that the same argument as in Proposition 1 shows that $f'_+(u) \geq -1$ for all u. Moreover, the same argument as in Proposition 1 shows that if $w(u) > \underline{w}$, then $f'(u) = -1$. Define u_2^* as the smallest payoff of the agent such that $f'_+(u) = -1$, i.e.,

$$u_2^* = \inf\{u : f'_+(u) = -1\}.$$

It is then clear that $f(u) = f(u_2^*) + (u_2^* - u)$ for all $u \geq u_2^*$, and the stated effort and and continuation payoffs support the payoff frontier.

Part (ii): When u is in the middle region. The definition of u_2^* implies that the agent's wage is given by $\underline{w}$. In addition, the agent chooses effort e to maximize

$$(1-\delta)(\underline{w} - c(e)) + \delta[p(e)u_h(u) + (1-p(e))u_l(u)],$$

and the first order condition with respect to e gives that

$$(1-\delta)c'(e) = \delta p'(e)(u_h(u) - u_l(u)),$$

and, thus,

$$u_h(u) = u_l(u) + \frac{1-\delta}{\delta}\frac{c'(e)}{p'(e)}.$$

Finally, the promise-keeping condition of the agent's utility gives that

$$\begin{aligned} u &= (1-\delta)(\underline{w} - c(e)) + \delta[p(e)u_h(u) + (1-p(e))u_l(u)] \\ &= (1-\delta)(\underline{w} - c(e)) + \delta[u_l(u) + p(e)\frac{1-\delta}{\delta}\frac{c'(e)}{p'(e)}], \end{aligned}$$

which implies that

$$u_l(u) = \frac{1}{\delta}\left(u - (1-\delta)(\underline{w} - c(e) + \frac{p(e)c'(e)}{p'(e)})\right).$$

This completes the proof. ■

While a full characterization of the employment dynamics is difficult for a general effort cost function, we are able to show that termination can still occur under the optimal relational contract for some cost functions. For example, let $p(e) = pe$ and consider the following piecewise linear cost function:

$$c(e) = \begin{cases} c_0 e & e \in [0, a] \\ c_0 a + c_1(e - a) & e \in (a, 1] \end{cases},$$

where $c_1 > c_0 > 0$. We assume that $c_1 < py$ so higher level of effort increases joint surplus.

COROLLARY A1: *If $\underline{w} \geq \underline{u}$, there exists an a^* such that the relationship terminates with positive probability for all $a \leq a^*$.*

Proof of Corollary A1: We show that when a is small enough, the PPE payoff frontier is essentially identical to that in the main model in the sense that if $e > 0$ is chosen then $e = 1$. To do this, we take two steps. First, we show that if $e > 0$ is chosen, then either $e = a$ or $e = 1$ is chosen. Second, we show that for sufficiently small a, only $e = 1$ is chosen.

In Step 1, we first show that if $e \in (0, a]$ is chosen, it is dominated by the choice of $e = a$. Suppose $e \in (0, a]$ is chosen, then the agent's IC is given by

$$u_h(e) - u_l(e) = \frac{1-\delta}{\delta}\frac{c_0}{p} \text{ for all } e \in (0, a].$$

The promise-keeping condition of the agent then implies that as long as $e > 0$, we have

$$\begin{aligned} u &= (1-\delta)(\underline{w} - c_0 e) + \delta(u_l(e) + pe(u_h(e) - u_l(e))) \\ &= (1-\delta)(\underline{w} - c_0 e) + \delta\left(u_l(e) + e\frac{1-\delta}{\delta}c_0\right). \end{aligned}$$

Solving the equation above, we have $u_l(e) = (u - (1-\delta)\underline{w})/\delta$. Notice that $u_l(e)$ is independent of e, and the IC constraint then immediately implies that $u_h(e)$ is also independent of e. Moreover, the expression for $u_l(e)$ implies that the smallest u to support effort $e \in (0, a]$ is given by $u_1^*(e) \equiv (1-\delta)\underline{w} + \delta\underline{u}$, which is also independent of e.

Next, notice that if u is supported by effort $e > 0$, the joint surplus $(u + f(u))$ is given by

$$\begin{aligned} &(1-\delta)(pey - c_0 e) \\ &+\delta(u_l + f(u_l) + (pe)(u_h + f(u_h) - u_l - f(u_l))). \end{aligned}$$

This term is increasing in e since $py - c_0 > 0$ and $u_h + f(u_h) \geq u_l + f(u_l)$. The second inequality follows because $f'(u) \geq -1$ by Proposition A3. It follows that if $e > 0$ is chosen, it is strictly dominated by the choice of $e = a$.

Next, we show that if $e \in (a, 1]$ is chosen, it is dominated by the choice of $e = 1$. In this case, then the agent's IC is given by

$$u_h(e) - u_l(e) = \frac{1-\delta}{\delta}\frac{c_1}{p}$$

for all $e \in (a, 1]$.

The promise-keeping condition then implies that

$$\begin{aligned} u &= (1-\delta)\left(\underline{w} - c_0 a - c_1(e-a)\right) + \delta\left(u_l(e) + pe\left(u_h(e) - u_l(e)\right)\right) \\ &= (1-\delta)\left(\underline{w} + (c_1 - c_0)a\right) + \delta u_l(e), \end{aligned}$$

and therefore,

$$u_l(e) = \frac{1}{\delta}\left(u - (1-\delta)\left(\underline{w} + (c_1 - c_0)a\right)\right).$$

Notice again that $u_l(e)$ is independent of e for all $e \in (a, 1]$, and therefore, so is $u_h(e)$. Moreover, the smallest u to support effort $e \in (a, 1]$ is given by

$$\begin{aligned} u_0(e) &\equiv (1-\delta)\left(\underline{w} - c_0 a - c_1(e-a)\right) + \delta\left(\underline{u} + \frac{1-\delta}{\delta}\frac{c_1}{p}e\right) \\ &= (1-\delta)\left(\underline{w} + (c_1 - c_0)a\right) + \delta\underline{u}, \end{aligned}$$

which is independent of e.

Next, notice that similar to the case of $e \in (0, a]$, if u is supported by effort $e \in (a, 1]$, the joint surplus is given by

$$\begin{aligned} &(1-\delta)\left(pey - c_0 a - c_1(e-a)\right) \\ &+\delta\left(u_l + f(u_l) + (pe)\left(u_h + f(u_h) - u_l - f(u_l)\right)\right). \end{aligned}$$

This term is again increasing in e since $py - c_1 > 0$ and $u_h + f(u_h) \geq u_l + f(u_l)$. It follows that if $e \in (a, 1]$ is chosen, it is strictly dominated by the choice of $e = 1$. This finishes Step 1.

Step 2: We now prove that for small enough a, only $e = 1$ is chosen. First note that $e = 0$ will not be chosen. The proof of this is identical to that in Lemma 3 (where we showed that $g(u) < f(u)$ when $\underline{w} \geq \underline{u}$) and is omitted here. Next, we show that $e = a$ will not be chosen for small enough a. Notice that if $e = a$ is chosen, the joint surplus is equal to

$$\begin{aligned} &S(a) = (1-\delta)\left(pay - c_0 a\right) \\ &+\delta\left(u_l(a) + f(u_l(a)) + (pa)\left(u_h(a) + f(u_h(a)) - u_l(a) - f(u_l(a))\right)\right). \end{aligned}$$

There are two cases to consider. In the first case, $u \geq (1-\delta)\left(\underline{w} + (c_1 - c_0)a\right) + \delta\underline{u}$. In this case, both $e = 1$ and $e = a$ are feasible. The joint surplus from choosing $e = 1$ is given by

$$S(1) = (1-\delta)(py - c_0 a - c_1(1-a))$$
$$+\delta\left(u_l(1) + f\left(u_l(1)\right) + p\left(u_h(1) + f\left(u_h(1)\right) - u_l(1) - f\left(u_l(1)\right)\right)\right).$$

Notice that as a goes to zero,

$$\lim_{a\to 0} S(a) = \delta\left(\frac{1}{\delta}\left(u - (1-\delta)\underline{w}\right) + f\left(\frac{1}{\delta}\left(u - (1-\delta)\underline{w}\right)\right)\right).$$

In contrast,

$$\begin{aligned}\lim_{a\to 0} S(1) &= (1-\delta)(py - c_1) + \delta\left(\frac{1}{\delta}\left(u - (1-\delta)\underline{w}\right) + f\left(\frac{1}{\delta}\left(u - (1-\delta)\underline{w}\right)\right)\right) \\ &\quad + \left(\frac{1-\delta}{\delta}\frac{c_1}{p} + f\left(\frac{1}{\delta}\left(u - (1-\delta)\underline{w}\right) + \frac{1-\delta}{\delta}\frac{c_1}{p}\right) - f\left(\frac{1}{\delta}\left(u - (1-\delta)\underline{w}\right)\right)\right).\end{aligned}$$

Notice that $f' \geq -1$, so

$$\left(\frac{1-\delta}{\delta}\frac{c_1}{p} + f\left(\frac{1}{\delta}\left(u - (1-\delta)\underline{w}\right) + \frac{1-\delta}{\delta}\frac{c_1}{p}\right) - f\left(\frac{1}{\delta}\left(u - (1-\delta)\underline{w}\right)\right)\right) \geq 0.$$

This implies that

$$\begin{aligned}&\lim_{a\to 0} S(1) - S(a) \\ \geq\ &(1-\delta)(py - c_1),\end{aligned}$$

and as a result, there exists a_1^* such that for all $a \leq a_1^*$, $S(1) > S(0)$. In other words, $e = a$ cannot be chosen for all $a \leq a_1^*$ here.

In the second case, $u < (1-\delta)\left(\underline{w} + (c_1 - c_0)a\right) + \delta\underline{u}$. In this case, $e = 1$ cannot be supported. However, $u_l(u) \to \underline{u}$ as a goes to 0. Therefore,

$$\lim_{a\to 0} S(a) = \delta\left(\underline{u} + f\left(\underline{u}\right)\right) = \delta\left(\underline{u} + \underline{v}\right) < \underline{u} + \underline{v},$$

which is a contradiction because the joint surplus cannot be smaller than the sum of the outside options. It follows that there exists a_2^* such that for $a \leq a_2^*$, $f(u)$ cannot be supported with $e = a$ for $u < (1-\delta)\left(\underline{w} + (c_1 - c_0)a_2^*\right) + \delta\underline{u}$.

Combining the two cases by having $a^* = \min\{a_1^*, a_2^*\}$, then only $e = 1$ can be chosen for all $a < a^*$, and the corollary is proved. ■