LEVELING UP WITHOUT BREAKING DOWN: STRATEGIES FOR SUCCESS UNDER STRESS*

JIN LI TAIQI LI ZHENGLI WANG

This paper explores strategies for achieving goals under stress. Higher effort results in faster progress but also increases the stress level, causing a higher breakdown rate. We characterize the optimal effort trajectory by using an exchangeability technique that compares different sequences of effort choices. The comparison reflects a trade-off between speed (for faster recovery) and risk (of breakdown). One consequence of this trade-off is that the individual should engage in preemptive stress management. Optimal strategies feature planned strategic retreats, such as cheat days in weight loss programs and rest days in marathon training.

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I. INTRODUCTION

This paper explores strategies for achieving goals under stress. We emphasize the dual effect of effort: While effort acts as a driving force toward the goal, it can also induce negative effects such as stress, causing the individual to break down and quit. We characterize the optimal strategy, using an *exchangeability* technique that compares different sequences of effort choices, that is, rest-then-sprint vs sprint-then-rest. The comparison of effort sequences reflects a trade-off between speed (for faster recovery) and risk (of breakdown). The resulting optimal strategies feature planned strategic retreats that manage stress preemptively.

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*Jin Li: Faculty of Business and Economics, The University of Hong Kong. E-mail: jli1@hku.hk. Taiqi Li: Faculty of Business and Economics, The University of Hong Kong. E-mail: taiqili@connect.hku.hk. Zhengli Wang: Faculty of Business and Economics, The University of Hong Kong. E-mail: wzl1@hku.hk. We thank Yeon-Koo Che, Florian Englmaier, Shuaizhang Feng, Mira Frick, Guido Friebel, Jie Gong, Carl Heese, Keri Hu, John Klopfer, Michael Kosfeld, Ruby Lu, Masaki Miyashita, Wing Suen, Balas Szentes, Can Urgun, Brian Viard, Yufu Wong, Yanhui Wu, Chen Zhao, and seminar participants at Fudan University, Goethe University of Frankfurt, Jinan University, University of Hong Kong, and University of Utah for helpful comments.

Consider an individual who studies French. As the individual practices French, it gradually becomes easier for him. But if the individual practices too much, it becomes unpleasant and causes him to quit. Successful learning then hinges on the balance between these two factors: the ease gained through practice and the potential aversion caused by overpracticing. The prevailing factor will determine the individual's success in reaching the goal.

The two forces are present in numerous settings, from mastering a musical instrument to reaching a weight loss goal, building the skill for running a race, and so on. Albert Hirschman (1958), in "The Strategy of Economic Development," write: *Somewhat like a person who decides in a fit of enthusiasm to learn a foreign language, a country that sets out on the road to development often does not realize the difficulties of the task ahead. As these difficulties appear, as it becomes clear that the price of development is a high one in terms of human suffering, social tensions, forced abandonment of traditional behavior and values, etc., "practice" may be reduced, contradictory and harmful economic policies are being adopted, and development will be slowed down and halted.* Many initiatives within organizations also experience similar patterns: the resources and work invested in the initiative, while helpful in pushing it forward, also fuel resistance that could stop the initiative midway. However, these resistances will die down if the initiative can persist and its merit gains enough acceptance.

The interaction of the two forces has been formally modeled by Simon (1954), who considers an individual learning French using the Berlitz method. Simon's model, however, assumes that the individual's behavior adjusts mechanically, meaning that the amount of effort exerted tomorrow is completely determined by today's effort and skill level. This implies that the initial effort of the individual completely determines the entire effort trajectory and also whether the goal can or cannot be achieved.

The purpose of this paper is to study the optimal strategy for reaching a goal when effort increases both skill and stress. Our model has two state variables: skill and stress level. Effort increases skill, and an individual receives a reward if his skill exceeds a particular level. Effort also affects the stress level, which determines the likelihood that the individual experiences a breakdown at each time point. Extensive psychological literature has shown that the adverse impact on the individual's physical state (e.g. executive and cognitive systems) accelerates as the individual's stress level increases (Aschbacher et al. (2013), Dickerson and Kemeny (2004), Chrousos (2009), Sapolsky (2004)), which may lead to a breakdown.

This is reflected in our model that the rate at which the individual quits accelerates as stress increases.

We also assume that the stress level increases if the individual's effort exceeds a threshold, and it decreases otherwise. Importantly, this threshold increases with the individual's existing skill. This assumption highlights an additional benefit of skill: an individual with higher skill can "practice" more.

We solve the individual's optimal strategies and show that they can be put in two distinct categories, depending on the level of the goal. The first category–the *modest-goal* case–occurs when the gap between the starting skill and the targeted skill is small. In this scenario, the optimal strategies are simple. The individual either *surge*, meaning that he always puts in the maximal effort. Or, if his initial stress level is high, the individual puts no effort until the stress level drops below a level, and then he surges.

The second category, which we label as the *ambitious-goal* case, occurs when the difference between the starting skill and the targeted skill is large. The ambitious-goal case poses a greater challenge for the individual, leading to more varied strategies. Depending on the initial stress level of the individual, there are three types of optimal strategies. First, when the stress level is low, the optimal strategy is again to surge.

Second, when the stress level is in the intermediate range, the optimal strategy takes the form of *HIIT*[1], a term that we use because a key feature of the optimal strategy resembles the popular exercise. The HIIT strategy has three phases. In the first phase, the individual exerts the maximum effort, leading to rapid skill improvement but also a rapid increase in stress levels. When stress becomes too high, the second phase starts. Here, the individual drops his effort to a low (but positive) level at the beginning of the phase and then increases it. However, the effort level remains relatively low and the stress level drops steadily. The second phase ends when the individual's skill is sufficiently high, so the final goal is close in sight. The strategy then transitions into the third phase, where the individual surges again.

The second phase of HIIT is a planned strategic retreat, which is akin to cheat days in weight management programs and rest days in marathon training regimes. Planned strategic retreat underscores the importance of pre-emptive stress management, contrasting the

[1]HIIT stands for high-intensity interval training. It features low-intensity movements between high-intensity movements.

myopic idea that stress is managed only when it reaches a sufficiently high level. The HIIT strategy implies that when the goal is ambitious, rather than waiting for the stress level to escalate, the individual should lower the stress level at an earlier stage. The advantage of doing so is that when the goal is eventually close in sight, the individual's stress level is sufficiently low so that he can surge.

Finally, when the initial stress level is high, the optimal strategy takes the form of *phased escalation*. This strategy again has three phases. Initially, the individual does not put in any effort, enabling the stress level to drop rapidly. Once the stress level drops below a skill-dependent level, the strategy transitions to the second phase, which can be viewed as a warm-up phase. Here, the effort gradually increases and the stress continues to decrease. Once the skill level reaches a high enough level, the third phase starts, and the effort jumps to the maximal level.

We solve for the optimal strategies by using an exchangeability argument that compares the payoff between sprint (maximal effort)-then-rest (zero effort) versus rest-then-sprint. This comparison reveals a trade-off between speed (for faster recovery) and risk (of breakdown). Sprint-then-rest has the cost of a higher risk of breakdown. However, it has a novel dynamic benefit of faster recovery (stress reduction) associated with a higher skill level.

Importantly, this benefit is higher when the individual's skill level is lower, implying that the individual is better off sprinting when his skill is lower. Because the individual is also better off sprinting when the goal is near in sight (when he has higher skills), two types of non-monotonicities arise. First, the individual's effort level can be non-monotone over time, as the HIIT strategy indicates. Second, for a fixed initial stress level, the optimal initial effort of the individual can be non-monotone at the goal level. The individual's initial action is to start small when the goal is at the intermediate level and to do a big push when both the goal levels are low or high.

Our model is related to several dynamic models that feature the negative effect of effort on future efforts. A closely related one is Urgun (2021), who assumes that the marginal cost of effort increases when he works and decreases when he is idle. Unlike our paper, Urgun (2021) is interested in how to allocate tasks among different agents.

Our model is also related to papers that model discrete jumps (in continuation payoffs) with the Poisson process; see, for example, Keller et al. (2005), Rosenberg et al. (2007), Keller and Rady (2010), Keller and Rady (2015), Strulovici (2010), Bonatti and Hörner

(2011), Klein and Rady (2011), Murto and Välimäki (2011), Board and Meyer-ter Vehn (2013), Guo (2016), Halac et al. (2017), Che and Mierendorff (2019), Zhong (2022), Che et al. (2023) and Liu and Wong (2023). In these models, Poisson arrivals reflect discrete jumps in beliefs. In contrast, our Poisson process describes the likelihood that the individual quits.

Finally, we characterize optimal strategies using a technique that relies on the exchange of sequences of actions. This exchangability technique is instrumental in solving HJB equations that involve more than one state variable. The exchangeability technique has been used in Li et al. (2023) to characterize the optimal experimentation strategy with multiple bandits. Unlike Li et al. (2023), the application of the exchangeability argument in this paper requires adjusting the length of the time intervals.

The rest of the paper is organized as follows. We setup the model in Section 2 and carry out the main analysis in Section 3. Section 4 concludes.

II. MODEL SETUP

To describe our setup, it is useful to first review Simon's Berlitz model.

A. Benchmark: Simon's Berlitz Model

Simon (1954) considers the example of an individual learning French through the Berlitz method. Let X_t be the amount of time (effort) the individual practices in period t. Simon's model has three key assumptions. First, as the individual practices it, his skill improves, and the corresponding difficulty level of the language decreases. In particular, denote the difficulty level at period t to be D_t. Simon assumes that it decreases logarithmically in effort with learning rate $a > 0$:

$$dD_t/dt = -aD_tX_t. \tag{1}$$

Second, it is assumed that at any level of difficulty, practice is pleasurable up to a certain point, and unpleasant beyond that point. This point is called the individual's *satiation level* $\bar{X}(D_t)$. The individual practices more when he finds it pleasurable (effort is below the satiation level) he practices less otherwise. Specifically, the individual's effort adjusts according to

$$dX_t/dt = -b(X_t - \bar{X}(D_t)). \tag{2}$$

Finally, the individual's satiation level $\bar{X}(D_t)$ decreases linearly with the difficulty of the problem. In other words, the higher the individual's skill level is, the higher his satiation level.

B. *Model Setup*

We now describe our setup. In every period t, the individual can choose to put in a certain amount of effort $X_t \in [0, x_U]$. Let D_t be the level of difficulty at period t. Same as Simon's model, we assume that the difficulty decreases logarithmically in effort:

$$dD_t/dt = -aD_tX_t. \tag{3}$$

Once the difficulty level diminishes to 1, the individual succeeds and receives a reward $\mathcal{R}$.

Next, to capture the idea that an increased effort level can create effort aversion in the future, we assume that effort can affect the individual's stress level, affecting the probability of quitting. Let Y_t denote the individual's stress level in period t. Consistent with the psychology literature that the individual's physical state (e.g. executive and cognitive systems) accelerates as the individual's stress level increases (Aschbacher et al. (2013), Dickerson and Kemeny (2004), Chrousos (2009), Sapolsky (2004)), we assume that the occurrence of a "quit" event follows a non-homogeneous Poisson process whose arrival rate is a non-negative, increasing, and convex function of Y_t (which reflects a greater combined physical and mental toll on the individual (Thoits (2010), O'Connor et al. (2021))). For simplicity, we assume that the rate is given by e^{Y_t}. This assumption allows us to characterize the closed-form solution of the individual's value function and derive comparative static results. In Section IV, we show that the main features of the optimal strategy are preserved as long as the arrival rate is log-convex in the stress level. When a "quit" event occurs, the individual fails and receives no rewards.

To model how the stress level Y_t changes, we assume that, as in Simon's model, the individual has a satiation level denoted as $\bar{X}(D_t)$. When the individual's effort exceeds the satiation level, his stress level increases. Otherwise, his stress level decreases. Specifically,

$$dY_t/dt = b[X_t - \bar{X}(D_t)]. \tag{4}$$

In other words, instead of directly affecting the effort level of tomorrow, effort affects the underlying stress level, and $b > 0$ is the rate at which the stress level changes. We follow Simon's illustration and assume that $\bar{X}(D_t) = k_1 - k_2 D_t$. This captures the idea that the individual's satiation level is higher when his skill level is higher (the difficulty level is lower).

Note that our modeling choice of the stress level allows the breakdown probability in each moment to depend on the entire past effort choices. In addition, it captures the idea that the marginal negative effect of effort is lower if the individual's skill level is higher. Our modeling choice dovetails with the idea that the marginal cost of effort may depend on all past effort choices. These effects are summarized by the stress level, which serves as a sufficient statistic for the impacts of all past effort choices.

To describe the individual's maximization problem formally, denote $\mathcal{N}_t$ as the number of "quit" event occurrences during $[0, t]$, and denote the terminal time T as

$$T = \inf\{t \mid D_t \leq 1 \text{ or } \mathcal{N}_t \geq 1\}. \tag{5}$$

The individual optimizes the following objective

$$\max_{X_t \in [0, x_U]} \mathbb{E}\left[e^{-\lambda T} \mathcal{R} \mathbf{1}_{\{D_T = 1\}}\right], \tag{6}$$

subject to equations (3) and (4). Here, $\lambda > 0$ represents the discount factor, and $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function.

We impose the following parameter restrictions to make the model interesting. First, we assume $k_1 - k_2 D_t > 0$ so that the individual's stress level always decreases when no effort is exerted. Second, we assume $k_1 - k_2 < x_U$ to ensure that the stress level always increases when maximal effort is exerted. Third, we assume that the parameter b exceeds a certain threshold, so that the stress level Y_t adjusts meaningfully: otherwise, the individual always exerts maximal effort in every period (see Lemma 7 in Appendix B for details).

We end the section with two remarks on our model. First, similar to Simon's model, our model is intentionally simple and stylized. In particular, we assume that the individual's sole objective is to develop the skill (quickly). We do not incorporate considerations of effort costs or utility loss associated with stress levels. Although these are obviously relevant features for many applications, the simplicity of our model gives the advantage of clearly highlighting the key features of the optimal strategy.

Second, unlike Simon's model, where the individual fails by gradually lowering his effort, our model features sudden failure through a Poisson process of quitting. Our model is therefore more relevant to examples where failures take the form of a breakdown.

III. ANALYSIS

In this section, we characterize the optimal strategy for skill development. We do so using a dynamic programming approach. In Section 3.1, we reformulate the problem as a dynamic programming problem. In Section 3.2, we describe our approach to solving the problem. Section 3.3 describes the optimal strategy.

A. *Preliminary Analysis and HJB*

The characterization of the optimal strategy requires a description of the entire effort sequence over time. As in many dynamic optimization problems, we can solve this problem with dynamic programming. Notice that in every period t, the current difficulty and stress level summarizes all information that is relevant to the individual's effort choice X_t. Therefore, the difficulty level D_t and the stress level Y_t are the state variables, and we can define the following value function[2]

$$V(D_t, Y_t) \triangleq \max_{X_t \in [0, x_U]} \mathbb{E}\left[e^{-\lambda T} \mathcal{R} \mathbf{1}_{\{D_T = 1\}} \,\middle|\, D_t, Y_t \right]. \tag{7}$$

We now derive, heuristically, the Hamilton-Jacobi-Bellman (HJB) equation associated with the value function. The HJB equation is a continuous-time analog of the Bellman equation in discrete-time models. Consider a small time interval $[t, t + \Delta t]$. Starting at any difficulty level D_t and stress level Y_t, the process may end (a "quit" event occurs) with probability $e^{Y_t}\Delta t + o(\Delta t)$ or proceed smoothly (no "quit" event occurs) with probability

[2] In the appendix, we show that the value function is well defined because the action set is compact.

$1 - e^{Y_t}\Delta t + o(\Delta t)$. Hence,

$$V(D_t, Y_t) = \max_{X_t \in [0, x_U]} e^{-\lambda \Delta t} \Big[\Big(1 - e^{Y_t}\Delta t + o(\Delta t)\Big) \cdot V(D_{t+\Delta t}, Y_{t+\Delta t}) + \Big(e^{Y_t}\Delta t + o(\Delta t)\Big) \cdot 0 \Big] + o(\Delta t)$$
$$= \max_{X_t \in [0, x_U]} \Big[1 - (\lambda + e^{Y_t})\Delta t\Big] \cdot V(D_{t+\Delta t}, Y_{t+\Delta t}) + o(\Delta t). \tag{8}$$

Applying Taylor expansion to $V(D_{t+\Delta t}, Y_{t+\Delta t})$ and sending $\Delta t \to 0$, we arrive at

$$0 = \max_{x \in [0, x_U]} \left\{ -(e^y + \lambda)V(D, y) - aDx\frac{\partial V(D, y)}{\partial D} + b[x - (k_1 - k_2 D)]\frac{\partial V(D, y)}{\partial y} \right\}, \tag{9}$$

where we substitute $D_t = D$, $Y_t = y$, $X_t = x$ and use the expressions of how the difficulty and the stress level change with effort in (3) and (4).

The right-hand side of equation (9) contains three terms. The first term, $-(e^y + \lambda)V(D, y)$, represents the usual discounting effect. Unlike most models, there is an additional e^y. This reflects the fact that the effective discount rate takes into account that the process breaks down with rate e^y. The second term, $-aDx\frac{\partial V(D,y)}{\partial D}$, involves the changes in the difficulty level of the problem: $-aDx$ reflects how the rate of the difficulty level changes, and $\partial V/\partial D$ is the marginal value (of the difficulty level) to the individual. The third term, $b[x - (k_1 - k_2 D)]\frac{\partial V(D,y)}{\partial y}$, relates the changes in the stress level: $b[x - (k_1 - k_2 D)]$ reflects how the rate of the stress level changes and $\partial V/\partial y$ is the marginal value of the stress level.

Equation (9) shows that the effort level x affects the right-hand side of HJB in two ways: on the one hand, an increase in x benefits the individual by reducing the difficulty level; on the other hand, an increase in x hurts him by reducing the rate at which the stress level decreases. The optimal choice of x, therefore, depends on the relative magnitude of these two effects.

B. *The Exchangeability Curve and Speed-Risk Trade-off*

The HJB is a partial differential equation with a maximization operator. The typical way to solve the equation is to first conjecture a solution and then verify it. Notice that the right-hand side of the HJB (equation (9)) is linear in x. The natural guess then is that the optimal solution is bang-bang. However, it turns out that this is not the case, which complicates the analysis.

To deal with the difficulty, we take a step back and take advantage of the sequential nature of the problem, that is, the individual chooses his effort one at a time. Our key solution technique relies on the following exchangeability argument. If under the optimal strategy, the individual, in two consecutive (short) time intervals, chooses to rest (no effort) first and sprint (maximal effort) next, his payoff should be weakly higher than an alternative effort trajectory where he first sprints and then rests (with the rest of the effort trajectory unchanged). In other words, the optimal strategy cannot be improved upon with the type of perturbation in which one exchanges the sequence of effort choices.

Before applying the exchangeability argument, we simplify the analysis by isolating a region in the space of state variables where the individual chooses the maximal effort all the time. Specifically, we first identify the points (D, y) where a) the optimal strategy specifies maximal effort at (D, y), and b) the optimal strategy specifies maximal effort along the trajectory following (D, y). We refer to these points as *surge points*, which describes the management practice in which individuals try their utmost until the task is completed.

LEMMA 1: *(Surge Region) There exists a decreasing function $g_1(\cdot)$ ("surge curve") such that (D, y) is a surge point if and only if $(D, y) \in \mathcal{M}_1 \triangleq \{(D, y); y \leq g_1(D)\}$.*

Lemma 1 shows that it is optimal for the individual to surge (keep making the highest effort level x_U) when the initial stress level is low and the difficulty level is low. This result follows from the fact that the individual's quitting probability is increasing and convex in the stress level. When the initial stress level is low, the cost of working–the increase in the quitting rate–is low, so the individual is better off putting in effort. Similarly, when the difficulty level is low, the time it takes to reach the goal is short when the individual keeps choosing the highest effort. This implies that the cumulative increase in the stress level (and therefore the quitting rate) will be small, again implying that it is optimal for the individual to surge.

Notice that, starting with any (D, y) in this region (which we call $\mathcal{M}_1$), any path traversed by the individual will continue in this region. That is, for all initial levels $(D, y) \in \mathcal{M}_1$, if we always select $X_t = x_U$, the trajectory of the whole process remains within $\mathcal{M}_1$. Otherwise, the initial point (D, y) fails to be a surge point.

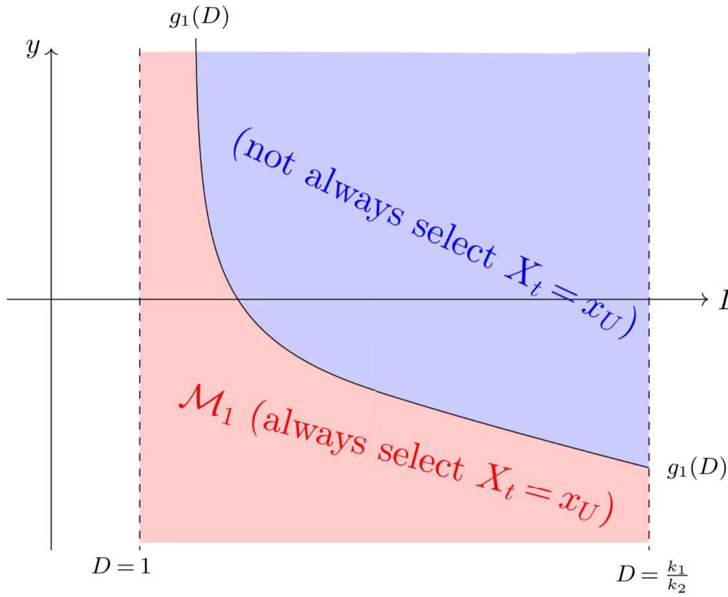


FIGURE 1.—Below $g_1(D)$: surge (always select $X_t = x_U$) is optimal. Above $g_1(D)$: surge is not optimal.

Lemma 1 shows that for $(D, y) \in \mathcal{M}_1$, the individual's optimal strategy is to always put in the highest effort. For points outside $\mathcal{M}_1$, the optimal strategy is more complicated, and we solve them using an exchangeability argument.

Specifically, suppose that the variable of initial state of the individual is represented by point A in Figure 2. Also, assume that the individual will arrive at point B at some point. The exchangeability argument considers two ways for the individual to reach point B. The first is represented by the blue curve, where the individual first chooses $X_t = x_U$ until the difficulty decreases by ΔD, then chooses $X_t = 0$ until the stress level decreases by Δy. We denote this curve by SR because the individual sprints first and then rests. The second is represented by the red curve, where the individual first selects $X_t = 0$ until the stress decreases by Δy, then chooses $X_t = x_U$ until the difficulty decreases by ΔD. We denote the red curve as RS: first rests, and then sprints.

We denote the individual's payoffs along the two curves as P_{SR} and P_{RS}, and the exchangeability argument compares these two payoffs. The comparison reflects a trade-off between risk (of breakdown) and speed (for recovery). For the individual, the disadvantage of choosing SR is that his stress level is higher than that of RS. This means that the cost of SR (compared to RS) is a higher risk of breaking down.

The advantage of choosing SR is that it takes less time to lower the stress level than the path RS. In other words, one benefit of sprinting first is that, by building up his skill first, the individual saves future recovery time (of stress). In other words, recovery speed is faster for individuals with higher skills. As an illustration of this point, consider the following example. An individual tries to get fit by starting a training exercise. In the first several sessions of exercise, the individual's muscles may get extremely sore and it takes a long time for him to recover. As the individual builds up his skill gradually, he still gets tired after the exercise, but it takes less time for him to recover. Therefore, building up the skill has the benefit of faster recovery.

The risk-speed trade-off determines when the individual should choose SR and when he should choose RS. The next lemma shows how the comparison depends on the difficulty and stress levels of the individual.

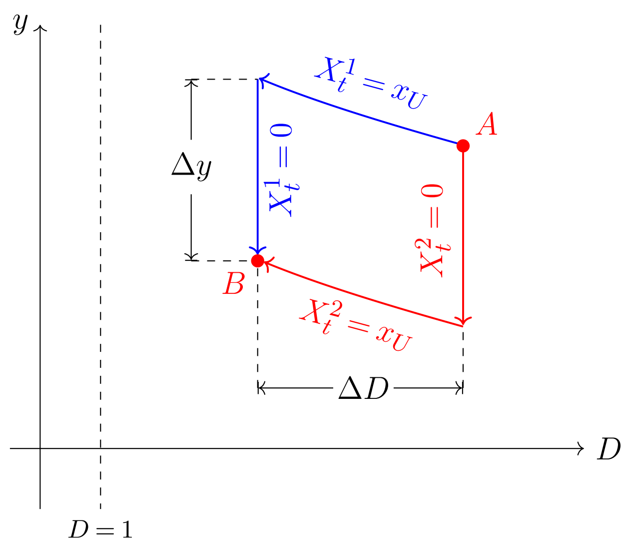


FIGURE 2.—The blue curve represents the trajectory when applying policy X_t^1, while the red curve represents the trajectory when applying X_t^2.

LEMMA 2: *(Exchangeability curve) When $\Delta D \to 0^+$ and $\Delta y \to 0^+$, there exists an increasing function $g_2(D)$ such that $P_{SR} = P_{RS}$ for the set of points $\{(D,y); y = g_2(D)\}$; $P_{SR} > P_{RS}$ for the set of points $\{(D,y); y < g_2(D)\}$; $P_{SR} < P_{RS}$ for the set of points $\{(D,y); y > g_2(D)\}$.*

Lemma 2 shows that the risk-speed trade-off favors sprinting (first) when the stress level is low and when the difficulty is high (so that the individual's skill level is low). To see why this is the case, note that when the stress level is low, the cost of sprinting first is small, favoring SR over RS. In addition, when difficulty is high (so that the skill level is low), the gain from faster recovery is greater. The bigger gain arises because the recovery time is longer for lower-skilled individuals. Using the earlier example, when the individual starts the exercise, it takes a long time to recover, and the reduction in recovery time from becoming more fit is significant, but when he is sufficiently fit, it takes him less time to recover, and the reduction in recovery time is smaller. In other words, the gain from reducing the recovery time is bigger when the difficulty level is high (the skill level is low), which favors SR over RS.

In Lemma 2, we denote $g_2(D)$ as *exchangeability curve*, which represents the function that at the set $\{(D,y); y = g_2(D)\}$, path SR and RS are equally preferable. We now use the exchangeability curve, together with the surge curve ($g_1(D)$ derived in Lemma 1), to partition the state space and derive the associated optimal actions.

C. *Optimal Strategies of Skill Development*

We now characterize the optimal strategies for skill development. Recall that $\mathcal{M}_1$ is the region below the surge curve. Now define $\mathcal{M}_2$ as the region that is above both the surge and exchangeability curve, and $\mathcal{M}_3$ as the region in between:

$$\mathcal{M}_2 \triangleq \{(D,y); y > \max\{g_1(D), g_2(D)\}\}, \tag{10}$$

$$\mathcal{M}_3 \triangleq \{(D,y); g_1(D) < y < g_2(D)\}. \tag{11}$$

LEMMA 3: *It is optimal for the individual to select $X_t = 0$ whenever $(D,y) \in \mathcal{M}_2$, and to select $X_t = x_U$ whenever $(D,y) \in \mathcal{M}_3$.*

Lemma 3 states that the individual always rests in $\mathcal{M}_2$ and sprints in $\mathcal{M}_3$. To give a heuristic argument for why the individual chooses to rest in $\mathcal{M}_2$, suppose to the contrary that there exists a point $(D^*, y^*) \in \mathcal{M}_2$ such that the individual sprints. By the definition of $\mathcal{M}_2$, SR is better than RS. This means that the individual cannot rest in the next moment because he would be better off switching the order and resting today. But if the individual sprints in the next moment, he will remain above the exchangeability curve, implying that he has to continue to sprint the moment after. In other words, once the individual starts to sprint in $\mathcal{M}_2$, he will sprint all the way. But this would imply that (D^*, y^*) is in region $\mathcal{M}_1$, which is a contradiction.

A similar argument shows why the individual must sprint in $\mathcal{M}_3$. Again, suppose to the contrary that the individual finds it optimal to rest in a point $(D^*, y^*) \in \mathcal{M}_3$. By the definition of $\mathcal{M}_3$, RS is better than SR. This means that the individual cannot sprint at the next moment. But if the individual rests in the next moment, he will remain below the exchangeability curve, implying that he has to continue to rest the moment after. But the individual cannot rest forever because then he will never reach the goal. This is a contradiction.

Lemma 1 and Lemma 3 describe the optimal actions for all points in the state space other than those on the exchangeability curve. To state the optimal strategies, note that because the surge curve is downward sloping and the exchangeability curve is upward sloping, let D_c be the unique difficulty level where the two curves intersect. This difficulty level divides the goals into two categories. We call a goal “modest" if the initial level of difficulty $D \leq D_c$ and “ambitious" if its initial level of difficulty $D > D_c$. We also partition $\mathcal{M}_1$ and $\mathcal{M}_2$ depending on whether the difficulty level exceeds $D \leq D_c$. We define $\mathcal{M}_{1m}$, $\mathcal{M}_{1a}$, $\mathcal{M}_{2m}$, and $\mathcal{M}_{2a}$ accordingly, where the subscript "*m*" represents modest, and "*a*" stands for ambitious. Figure 3 provides an illustration of the regions.

THEOREM 1: *(Characterization of Optimal Policy). Given the initial states (D, y), the optimal policy has the following structures.*

I (Modest Goal). Suppose $D \leq D_c$:

*(i) **[Surge]** If $(D, y) \in \mathcal{M}_{1m}$, the optimal policy is to surge (always select $X_t = x_U$).*

*(ii) **[Rest then Surge]** If $(D, y) \in \mathcal{M}_{2m}$, the optimal policy has two phases*

$$X_t = \begin{cases} 0 & , t \in [0, \frac{y - g(D)}{b(k_1 - k_2 D)}), \\ x_U & , \textit{otherwise}. \end{cases} \tag{12}$$

II (Ambitious Goal). Suppose $D > D_c$:

*(i) **[Surge]** If $(D, y) \in \mathcal{M}_{1a}$, the optimal policy is to surge (always select $X_t = x_U$).*

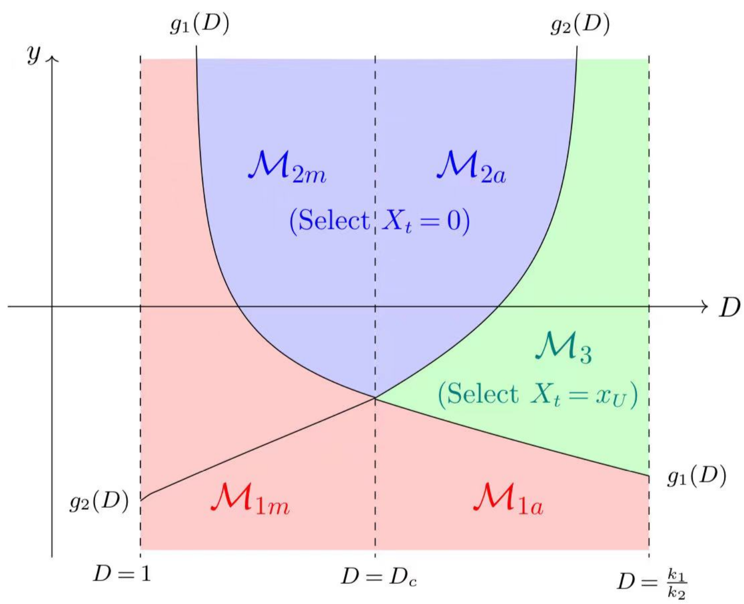


FIGURE 3.—$g_1(D)$ and $g_2(D)$ intersects when $D = D_c$. It is optimal to select $X_t = 0$ in $\mathcal{M}_2$ (i.e. $\mathcal{M}_{2m} \bigcup \mathcal{M}_{2a}$) and to select $X_t = x_U$ in $\mathcal{M}_3$.

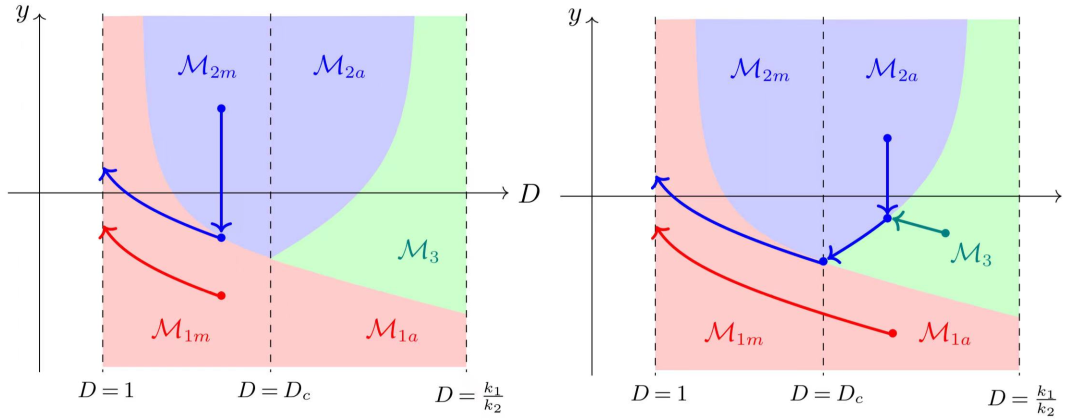
(ii) ***[Phased Escalation]*** *If $(D, y) \in \mathcal{M}_{2a}$, there exists $t_1, t_2 \geq 0$ such that the optimal policy has three phases*

$$X_t = \begin{cases} 0 & , t \in [0, t_1), \\ \frac{bk_1}{b+1} - k_2 D_t & , t \in [t_1, t_1 + t_2), \\ x_U & , \textit{otherwise}. \end{cases} \tag{13}$$

(iii) ***[HIIT]*** *If $(D, y) \in \mathcal{M}_3$, there exists $t'_1, t'_2 \geq 0$ such that the optimal policy has three phases*

$$X_t = \begin{cases} x_U & , t \in [0, t'_1), \\ \frac{bk_1}{b+1} - k_2 D_t & , t \in [t'_1, t'_1 + t'_2), \\ x_U & , \textit{otherwise}. \end{cases} \tag{14}$$

Figure 4 illustrates the two different cases of Theorem 1. The left panel (Figure 4(a)) describes the case where the goal is modest. In this case, the optimal policy can take two



(a) Modest Goals. (b) Ambitious Goals.

FIGURE 4.—Trajectory of optimal policy with initial states (D, y) in different regions.

forms, depending on whether the initial stress level y is low or high relative to the surge curve $g_1(D)$. When the initial stress level y is relatively low, the optimal policy, illustrated by the red arrow, is to surge by consistently selecting the maximal effort $x = x_U$. Doing so allows the individual to reach the goal as quickly as possible without unduly increasing the stress level.

When the initial stress level y is relatively high, the surge is no longer the optimal policy because doing so will result in an excessively high stress level. The optimal policy, illustrated by the blue arrow, is to rest first until the stress level falls below the surge curve. Once the individual is sufficiently rested, he then sprints.

The right panel (Figure 4(b)) describes the case where the goal is ambitious. In this case, the optimal policy can take three forms. The first one, similar to the modest-goal case, is to surge. This happens again when the initial stress level is below the surge curve $g_1(D)$.

When the initial stress level is in the intermediate range (that is, when the stress level is between the surge curve $g_1(D)$ and the exchangeability curve $g_2(D)$), the optimal policy takes the form of "HIIT". In particular, the individual starts with maximum effort. Doing so allows fast skill development, but it also increases stress rapidly. At some point, as illustrated by the green arrow, the stress level reaches the exchangeability curve. At this point, the individual immediately reduces his effort to a low (but positive) effort level. Once the effort is reduced, the individual adjusts his effort gradually so that the resulting trajectory stays on the exchangeability curve. Both the difficulty level and the stress level decrease steadily. However, the individual effort gradually increases. Finally, once the difficulty level drops to D_c, the individual transitions back to the maximal effort level.

When the initial stress level is very high, that is, the initial stress level y exceeds the threshold $g_2(D)$, the optimal policy takes the form of Phased Escalation. The individual initially rests to reduce the stress level. When the stress level is reduced to $g_2(D)$, the individual instantly increases his effort to a moderate level of effort. As in the "HIIT" case, the individual increases his effort steadily so that the resulting trajectory stays on the exchangeability curve $g_2(D)$. And once the difficulty level drops to D_c, the individual instantly increases his effort to the maximal level $x = x_U$.

D. Implications of the Optimal Strategies

We now summarize the key implications derived from our analysis, offering insights on how individuals should choose their effort trajectory to best develop their skills.

Planned Strategic Retreat as Pre-emptive Stress Management. Our optimal strategies highlight the importance of a planned strategic retreat, which occurs when the individual has an ambitious goal and a high initial stress level, that is, in the region that calls for HIIT strategy. In this case, the individual's effort drops following an initial sprint. The advantage of the planned strategic retreat is that it manages stress preemptively. As a result, the individual's stress level is sufficiently low so that he can surge when the goal is within reach. In other words, by managing stress levels in advance, the individual avoids a rest-and-surge scenario when the final goal is in sight.

In addition to calling for pre-emptive stress management, our strategies also prescribe how to carry it out. First, the individual's effort is not zero during the strategic retreat. That is, he continues to build his skill during the strategic retreat, albeit at a low speed. Second, the individual gradually increases his effort during the retreat. Therefore, the stress reduction rate decreases, and the skill-building rate increases. Third, the effort level is strictly bounded away from the maximal level. Consequently, the individual's effort level experiences a discontinuous increase over the strategic retreat ends.

Big Push vs Starting Small. Our model also provides guidance on how an individual should initiate his journey toward the goal. There are two distinct approaches: "Big Push" and "Starting Small". The Big-Push strategy involves initially putting in the maximum effort. It occurs when the goal level is either very high or very low. For intermediate goal levels, the individual starts small. He waits for the stress level to drop and gradually increases his effort.

This implies that under the optimal strategies, the initial effort level is U-shaped at the goal level, suggesting that the effect of the goal level can be subtle. When the goal level is very low, Big Push has the advantage of reaching the goal as soon as possible. However, when the goal distance increases to an intermediate level, Big Push would raise the stress level to such a high level at some point that the breakdown becomes likely. The optimal strategy then calls for starting small. When the goal level is very high, starting small becomes suboptimal because doing so will take too long to reach the goal: the individual gains by frontloading some of the efforts. That is, the individual should make a big push at the very beginning. But rather than sprinting all the way, he plans a strategic retreat midway to manage the stress.

Strategic Choice and Stress Factors. Our model shows that the choice of optimal strategies depends on many factors. Naturally, the goal's distance is a crucial factor. But individual characteristics also matter. For example, for the same goal distance, an individual who learns fast may sprint all the way. In contrast, an individual who learns slowly will find the goal ambitious and plan a strategic retreat midway. However, learning speed aside, our model also shows that stress matters a lot in determining the optimal strategy.

Our model highlights three types of stress-related characteristics. First, how fast does the stress level increase in effort? Second, what is the individual's baseline stress tolerance level? Finally, what is the rate at which the stress tolerance level changes with skill? The choice of optimal strategy requires taking these three types of stress characteristics into account. Specifically, we can show that when the stress tolerance level increases faster in skill, the HIIT strategy is more likely (in the sense that the exchangeability curve moves up). The faster build-up in tolerance level then calls for both a Big Push initially and a planned strategic retreat.

IV. GENERALIZATION OF THE EXCHANGEABILITY TECHNIQUE

In this section, we extend the exchangeability technique by allowing the arrival rate of the 'quit' event to be a general function of the individual's stress level. Specifically, the quit intensity at each time t is given by $h(Y_t)$, where $h(\cdot)$ is a general stress-dependent function (rather than the specific form e^{Y_t}). In addition to the non-negativity and monotonicity assumptions on $h(\cdot)$ discussed in §III.*B*, we further assume that $h(\cdot)$ is log-convex, which implies that the rate at which the individual quits accelerates as stress increases. In the following, we will refer to $h(\cdot)$ as the stress function.

With the introduction of a general stress function, we can similarly identify a region in the state space where it is optimal for the individual to consistently exert maximal effort. Specifically, we define *surge points* (D, y), where: (a) the optimal strategy prescribes maximal effort at (D, y), and (b) the optimal strategy continues to prescribe maximal effort along the trajectory following (D, y).

LEMMA 4: *(Surge Region) When the stress function* $h(\cdot)$ *is log-convex, there exists a decreasing function* $g_1^{(h)}(\cdot)$ *(the "surge curve") such that* (D, y) *is a surge point if and only if* $(D, y) \in \mathcal{M}_1^{(h)} \triangleq \{(D, y); y \leq g_1^{(h)}(D)\}$.

For points outside $\mathcal{M}_1^{(h)}$, we solve them using a similar exchangeability argument. For given $\Delta D, \Delta y > 0$, we compare two paths: SR and RS. In the SR path, the individual first selects $X_t = x_U$ until difficulty decreases by ΔD, then switches to $X_t = 0$ until the stress decreases by Δy. In the RS path, the individual first selects $X_t = 0$ until the stress decreases by Δy, and then selects $X_t = x_U$ until the difficulty decreases by ΔD. A similar derivation comparing the relative benefits of these two paths (P_{SR} versus P_{RS}) again highlights the trade-off between risk and speed, as shown in the following lemma.

LEMMA 5: *(Exchangeability Curve) When the stress function* $h(\cdot)$ *is log-convex, as* $\Delta D \to 0^+$ *and* $\Delta y \to 0^+$*, there exists an increasing function* $g_2^{(h)}(D)$ *such that* $P_{SR} = P_{RS}$ *for points where* $y = g_2^{(h)}(D)$*;* $P_{SR} > P_{RS}$ *for points where* $y < g_2^{(h)}(D)$*; and* $P_{SR} < P_{RS}$ *for points where* $y > g_2^{(h)}(D)$.

As a result, we can characterize the optimal strategies for skill development analogously. Define $\mathcal{M}_2^{(h)}$ as the region above both the surge and exchangeability curves, and $\mathcal{M}_3^{(h)}$ as the region between them:

$$\mathcal{M}_2^{(h)} \triangleq \{(D, y); y > \max\{g_1^{(h)}(D), g_2^{(h)}(D)\}\}, \tag{15}$$

$$\mathcal{M}_3^{(h)} \triangleq \{(D, y); g_1^{(h)}(D) < y < g_2^{(h)}(D)\}. \tag{16}$$

LEMMA 6: *When the stress function $h(\cdot)$ is log-convex, it is optimal for the individual to select $X_t = 0$ whenever $(D, y) \in \mathcal{M}_2^{(h)}$, and to select $X_t = x_U$ whenever $(D, y) \in \mathcal{M}_3^{(h)}$.*

Since the surge curve is downward sloping and the exchangeability curve is upward sloping, we can similarly denote D_c as the unique difficulty level where the two curves intersect and classify goals as "modest" or "ambitious" based on whether the initial difficulty level is $D \leq D_c$ or $D > D_c$. We similarly partition $\mathcal{M}_1^{(h)}$ and $\mathcal{M}_2^{(h)}$ based on whether $D \leq D_c$, defining regions $\mathcal{M}_{1m}^{(h)}, \mathcal{M}_{1a}^{(h)}, \mathcal{M}_{2m}^{(h)}$, and $\mathcal{M}_{2a}^{(h)}$ (see Figure 3), and we can show that the optimal policy preserves the same structure as in Theorem 1.

THEOREM 2: *(Characterization of Optimal Policy under General Stress Function). When the stress function $h(\cdot)$ is log-convex, given the initial states (D, y), the optimal policy has the following structures.*

I (Modest Goal). Suppose $D \leq D_c$:

*(i) **[Surge]** If $(D, y) \in \mathcal{M}_{1m}^{(h)}$, the optimal policy is to surge (always select $X_t = x_U$).*

*(ii) **[Rest then Surge]** If $(D, y) \in \mathcal{M}_{2m}^{(h)}$, the optimal policy has two phases*

$$X_t = \begin{cases} 0 & , t \in [0, \frac{y - g_1^{(h)}(D)}{b(k_1 - k_2 D)}), \\ x_U & , \textit{otherwise}. \end{cases} \tag{17}$$

II (Ambitious Goal). Suppose $D > D_c$:

*(i) **[Surge]** If $(D, y) \in \mathcal{M}_{1a}^{(h)}$, the optimal policy is to surge (always select $X_t = x_U$).*

*(ii) **[Phased Escalation]** If $(D, y) \in \mathcal{M}_{2a}^{(h)}$, there exists $t_1, t_2 \geq 0$ such that the optimal policy has three phases*

$$X_t = \begin{cases} 0 & , t \in [0, t_1), \\ \frac{b(k_1 - k_2 D_t)[b(k_1 - k_2 D_t) h''(Y_t) - k_2 D_t h'(Y_t)]}{b^2 (k_1 - k_2 D_t) h''(Y_t) + [\lambda + h(Y_t)] k_2 D_t} & , t \in [t_1, t_1 + t_2), \\ x_U & , \textit{otherwise}. \end{cases} \tag{18}$$

(iii) ***[HIIT]*** *If* $(D,y)\in\mathcal{M}_3^{(h)}$*, there exists* $t_1',t_2'\geq 0$ *such that the optimal policy has three phases*

$$X_t=\begin{cases} x_U & ,\ t\in[0,t_1'),\\ \frac{b(k_1-k_2D_t)[b(k_1-k_2D_t)h''(Y_t)-k_2D_th'(Y_t)]}{b^2(k_1-k_2D_t)h''(Y_t)+[\lambda+h(Y_t)]k_2D_t} & ,\ t\in[t_1',t_1'+t_2'),\\ x_U & ,\ \textit{otherwise}.\end{cases} \tag{19}$$

V. CONCLUSION

This paper studies the optimal strategy for reaching a goal in which effort increases both skill and stress. We adopt an exchangeability argument to characterize the optimal strategy. The results show that the optimal strategy depends both on the individual characteristic and on the targeted goal. When the goal is modest, the individual either sprints all the way (surge) or rests first and then surges. When the goal is ambitious, the individual may still surge, but the rest-then-surge is no longer optimal. Instead, the individual may carry out phased escalation, or he does HIIT: sprint initially, followed by a planned strategic retreat, and sprint again. The planned strategic retreat highlights the importance of preemptive stress management, which sets the stage for the later surge.

The simplicity of the model allows it to be extended in the future in several directions. For example, the goal of the individual is given exogenously, and one may study how best to set the goal. Second, there is only one goal in the model, and a natural follow-up is to consider how the individual should dynamically allocate effort (and leisure) across multiple goals. Finally, we may apply the model to an organizational setting, where the increase in resistance of a project (which corresponds to the increase in stress level in this model) depends on the actions of the other players. Understanding this may help organizations implement changes more effectively.

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APPENDIX A: SCALING OF a

For the original problem formulation (3) and (4), we can perform a change of scale to make $a = 1$. This change of scale, described below, allows us to only consider the effect of b, k_1, k_2, and x_U, therefore simplifying the subsequent analysis.

Recall the evolution equation (3). If we define $\tilde{X}_t = aX_t$, then we have

$$dD_t/dt = -D_t\tilde{X}_t,$$

and correspondingly, we define $\tilde{x}_U = ax_U$. Then we define $\tilde{b} = \frac{b}{a}$, $\tilde{k_1} = ak_1$, and $\tilde{k_2} = ak_2$. We now have

$$dY_t/dt = \tilde{b}(\tilde{X}_t - (\tilde{k_1} - \tilde{k_2}D_t)).$$

Therefore, in our problem formulation, we can assume $a = 1$ without loss of generality. Throughout the remainder of the Appendix, we will proceed under the assumption that $a = 1$, unless specified otherwise.

APPENDIX B: PROOF OF LEMMA 1, 2 AND 3

In this section, we establish Lemma 1, 2 and 3. To achieve this, we first introduce two useful functions $f(D)$ (defined in (20)) and $\Gamma(D)$ (defined in (27)) and show that they satisfy certain properties in a series of Lemmas. In particular, Lemma 7 and 8 shows the properties of $f(D)$ and Lemma 9 shows the properties of $\Gamma(D)$. After that, using the results of Lemma 7, 8, and Lemma 9, we prove Lemma 1, 2 and 3.

We start by defining the function

$$f(D) = x_U + bZ(D)(k_1 - k_2D)e^{b(x_U-k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D}, \tag{20}$$

where

$$Z(D) = \int_1^D -\frac{1}{s}e^{-b(x_U-k_1)\frac{\ln s}{x_U} - \frac{bk_2}{x_U}s}ds.$$

LEMMA 7: *There exists a threshold b_+ such that, if $b \le b_+$, then $f(D) \ge 0$ for all $D \in [1, \frac{k_1}{k_2})$; $b > b_+$, then there exists $D \in [1, \frac{k_1}{k_2})$ such that $f(D) < 0$.*

Proof of Lemma 7. We can do the following transformations:

$$
\begin{aligned}
f(D,b) &= x_U + bZ(D)(k_1 - k_2 D)e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D} \\
&= x_U - (k_1 - k_2 D)\int_1^D \frac{b}{s} e^{\frac{b(x_U-k_1)}{x_U}(\ln D - \ln s) + \frac{bk_2}{x_U}(D-s)} ds
\end{aligned}
$$

When $b = 0$, obviously we have $f(D) = x_U$, i.e. $f(D) > 0$ for all $D \in [1, \frac{k_1}{k_2})$. When $b \to +\infty$, $f(D,b) \to -\infty$ for any $D \in (1, \frac{k_1}{k_2})$. Also, note that

$$
\begin{aligned}
\frac{\partial f(D,b)}{\partial b} = -(k_1 - k_2 D)\int_1^D \frac{1}{s}\left(\frac{b(x_U - k_1)}{x_U}(\ln D - \ln s) + \frac{bk_2}{x_U}(D - s) + 1\right) \\
e^{\frac{b(x_U-k_1)}{x_U}(\ln D - \ln s) + \frac{bk_2}{x_U}(D-s)} ds, \qquad (21)
\end{aligned}
$$

so $\frac{\partial f(D,b)}{\partial b}$ has the sign opposite of to $b(x_U - k_1)(\ln D - \ln s) + bk_2(D - s) + x_U$. Since we require $x_U > k_1 - k_2$ and $s \in [1, D]$, we have

$$
\begin{aligned}
b(x_U - k_1)(\ln D - \ln s) + bk_2(D - s) &> b(k_1 - k_2 - k_1)(\ln D - \ln s) + bk_2(D - s) \\
&= bk_2(D - s - \ln D + \ln s) \qquad (22)
\end{aligned}
$$

Since we have

$$
\frac{\partial}{\partial s}(D - s - \ln D + \ln s) = \frac{1}{s} - 1 \le 0
$$

for $s \in [1, D]$, and when $s = D$, $D - s - \ln D + \ln s = 0$. Hence for $s \in [1, D]$, we always have $D - s - \ln D + \ln s \ge 0$. Plugging it in equation (22) yields

$$
b(x_U - k_1)(\ln D - \ln s) + bk_2(D - s) + x_U > x_U. \qquad (23)
$$

Plugging the inequality (23) into equation (21) yields $\frac{\partial f(D,b)}{\partial b} < 0$, i.e. $f(D,b)$ is monotone in b. Hence there must exists a threshold b_+, such that, if $b \le b_+$, then $f(D) \ge 0$ for all $D \in [1, \frac{k_1}{k_2})$; $b > b_+$, then there exists $D \in [1, \frac{k_1}{k_2})$ such that $f(D) < 0$. □

In the remainder of the appendix, we always assume that $b > b_+$. Such assumption is without loss of generality. This is because if the assumption does not hold, then surging is always optimal for any (D, y).

LEMMA 8: *$f(D)$ has exactly two zeros D_L and D_U with $1 < D_L < \frac{bk_1}{(b+1)k_2} < D_U < \frac{k_1}{k_2}$ such that we have $f(D) < 0$ over (D_L, D_U) and $f(D) \geq 0$ over $[1, \frac{k_1}{k_2}) \setminus (D_L, D_U)$.*

Proof of Lemma 8. By Lemma 7, $b > b_+$ indicates there exists $D \in [1, \frac{k_1}{k_2})$ such that $f(D) < 0$. Also note that

$$f(1) = x_U,\ f(\frac{k_1}{k_2}) = x_U,$$

so by continuity, there must exist at least two zeros D_L, D_U that complies $D_L < D_U$, $f'(D_L) < 0$, and $f'(D_U) > 0$. We denote the zeros as D_i, where $i = 1, 2, 3, \dots$ and we investigate the derivatives at the zeros. Take derivative, we have

$$\begin{aligned} f'(D) = &-\frac{b}{D}(k_1 - k_2 D) - bk_2 Z(D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D} \\ &+ \frac{b(x_U - (k_1 - k_2 D))}{Dx_U} bZ(D)(k_1 - k_2 D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D}. \end{aligned} \tag{24}$$

Notice that, for zeros D_i, by definition we have

$$x_U + bZ(D_i)(k_1 - k_2 D_i) e^{b(x_U - k_1)\frac{\ln D_i}{x_U} + \frac{bk_2}{x_U}D_i} = 0,$$

so we can substitute $bZ(D_i)(k_1 - k_2 D_i) e^{b(x_U - k_1)\frac{\ln D_i}{x_U} + \frac{bk_2}{x_U}D_i}$ by $-x_U$ to equation (24), and obtain

$$f'(D_i) = \frac{(b+1)k_2 D_i - bk_1}{D_i(k_1 - k_2 D_i)} x_U \tag{25}$$

Let D_L be the smallest D_i that complies $f'(D_i) < 0$, we have $D_L \in [1, \frac{bk_1}{(b+1)k_2})$; let D_U be the largest D_i that complies $f'(D_i) > 0$, we have $D_U \in (\frac{bk_1}{(b+1)k_2}, \frac{k_1}{k_2})$. Then we cannot have any zeros in the interval $D \in [1, D_L)$. This is because D_L is the smallest D_i that complies $f'(D_i) < 0$, and by equation (25), $f'(D_i) \geq 0$ requires $D_i \geq \frac{bk_1}{(b+1)k_2} > D_L$. Similarly, we cannot have any zeros in the interval $D \in [D_U, \frac{k_1}{k_2})$. Recall that $f(1) = f(\frac{k_1}{k_2}) = x_U > 0$. Consequently, we have $f(D) \geq 0$ over $[1, \frac{k_1}{k_2}) \setminus (D_L, D_U)$.

Next we need to prove that there are no zeros in the interval $D \in (D_L, D_U)$. We suppose by contradiction that there exists a zero $D_p \in (D_L, D_U)$ and we discuss the following scenarios.

Scenario 1: $f'(D_p) < 0$. Note that $\lim_{D \to D_L^+} f(D) < 0$ and $\lim_{D \to D_p^-} f(D) > 0$, which implies there must exist another zero D_q over (D_L, D_p) such that $f'(D_q) > 0$. However, by equation (25), $f'(D_p) < 0$ implies that $D_p < \frac{bk_1}{(b+1)k_2}$, and $f'(D_q) > 0$ implies that $D_q > \frac{bk_1}{(b+1)k_2}$, i.e. $D_q > D_p$, which leads to contradiction.

Scenario 2: $f'(D_p) > 0$. Note that $\lim_{D \to D_p^+} f(D) > 0$ and $\lim_{D \to D_U^-} f(D) < 0$, which implies there must exist another zero D_q over (D_p, D_U) such that $f'(D_q) < 0$. However, by equation (25), $f'(D_p) > 0$ implies that $D_p > \frac{bk_1}{(b+1)k_2}$, and $f'(D_q) < 0$ implies that $D_q < \frac{bk_1}{(b+1)k_2}$, i.e. $D_q < D_p$, which leads to contradiction.

Scenario 3: $f'(D_p) = 0$. Then by equation (25) we know that $D_p = \frac{bk_1}{(b+1)k_2}$. Note that, by equation (25),

$$\lim_{D \to D_p^-} f'(D) < 0, \quad \lim_{D \to D_p^+} f'(D) > 0,$$

which indicates $f(D_p)$ is a local minimum, i.e. $\lim_{D \to D_p^-} f(D) > 0$ and $\lim_{D \to D_p^+} f(D) > 0$, so there must exist at least a zero D_j over (D_L, D_p) such that $f'(D_j) > 0$ and another zero D_q over (D_p, D_U) such that $f'(D_q) < 0$. However, by equation (25), $f'(D_j) > 0$ implies that $D_j > \frac{bk_1}{(b+1)k_2} = D_p$, and $f'(D_q) < 0$ implies that $D_q < \frac{bk_1}{(b+1)k_2} = D_p$, which leads to contradiction.

Hence there are no zeros in the interval $D \in (D_L, D_U)$, and $f(D) < 0$ over (D_L, D_U). $\square$

Next, we define the following function

$$\Gamma(D) = \begin{cases} \infty, \text{ if } f(D) \geq 0 \\ \ln\left(\frac{-\lambda x_U}{f(D)}\right), \text{ otherwise} \end{cases} \tag{26}$$

This function has the following properties.

LEMMA 9: *The function $\Gamma(D)$ has following properties:*
(a). The equation

$$\Gamma(D) = \ln\left(-\frac{\lambda k_2 D}{k_2(b+1)D - bk_1}\right), \tag{27}$$

has a unique solution $D = D_c$ where $D_c \in (D_L, \frac{bk_1}{(b+1)k_2})$, and we have

$$\Gamma'(D_c) = -\frac{bk_2}{x_U} - \frac{b(x_U - k_1)}{D_c x_U}.$$

(b). For $D_L < D < D_c$, we always have

$$\Gamma'(D) < -\frac{bk_2}{x_U} - \frac{b(x_U - k_1)}{D x_U}.$$

Proof of Lemma 9. We first prove part (a). Note that

$$\Gamma'(D) = \frac{f(D)}{-\lambda x_U} \cdot \frac{\lambda x_U f'(D)}{(f(D))^2} = -\frac{f'(D)}{f(D)}.$$

We can transform the equation (27) as follows:

$$-x_U = k_2 D Z(D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U} D},$$

where the right-hand side is monotone in D. Note that when $D = 1$, RHS is 0 which is larger than LHS; when $D = \frac{bk_1}{(b+1)k_2}$, we have $b(k_1 - k_2 D) = k_2 D$, so we obtain that

$$k_2 D Z(D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U} D} = b(k_1 - k_2 D) Z(D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U} D} < -x_U.$$

The inequality holds because, by Lemma 8, $f(D) < 0$ when $D = \frac{bk_1}{(b+1)k_2}$. Thus, there must exist a unique $D = D_c$. Then we prove $\Gamma'(D_c) = -\frac{bk_2}{x_U} - \frac{b(x_U - k_1)}{D_c x_U}$. Or equivalently, we should prove

$$\frac{f'(D_c)}{f(D_c)} = \frac{bk_2}{x_U} + \frac{b(x_U - k_1)}{D_c x_U}. \tag{28}$$

Since

$$f'(D_c) = \frac{b(x_U - (k_1 - k_2 D_c))}{D_c}\left(1 - \frac{b(k_1 - k_2 D_c)}{k_2 D_c}\right),$$

and

$$f(D_c) = x_U \left(1 - \frac{b(k_1 - k_2 D_c)}{k_2 D_c}\right),$$

we have equation (28) holds. Hence $D = D_c$ is the unique solution.

Then we prove part (b). We consider another function

$$h_1(x) = -\frac{-\frac{b}{D}(k_1 - k_2 D) - bk_2 x + \frac{b(x_U - (k_1 - k_2 D))}{D x_U} b(k_1 - k_2 D)x}{x_U + b(k_1 - k_2 D)x}$$

and note that, when $x = Z(D)e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D}$, we have $h(x) = \Gamma'(D)$. Also, note that when $x = -\frac{x_U}{k_2 D}$, we have $h_1(x) = -\frac{bk_2}{x_U} - \frac{b(x_U - k_1)}{D x_U}$. Take derivative, we have

$$h_1'(x) = -\frac{b x_U (bk_1 - (b+1)k_2 D)}{D(x_U + b(k_1 - k_2 D)x)^2}.$$

Recall that $D_c < \frac{bk_1}{(b+1)k_2}$, so for any $D \in (D_L, D_c)$, $h_1'(x) < 0$. Since $D < D_c$, we have $Z(D)e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D} > -\frac{x_U}{k_2 D}$, and therefore we have $\Gamma'(D) < -\frac{bk_2}{x_U} - \frac{b(x_U - k_1)}{D x_U}$. □

We are now ready to prove Lemma 1, 2 and 3.

A. *Proof of Lemma 1.*

Proof of Lemma 1. In this proof, we consider the following two strategies:

- Policy 1: Always select $X_t = x_U$.
- Policy 2: First select $X_t = 0$ and let Y_t decrease by Δy, then always select $X_t = x_U$.

Suppose the initial point is (D, y) and we take policy 1 as our standard. Then the strength of policy 2 is the relatively lower risk in the sprint ($X_t = x_U$) stage, but it suffers from the time disadvantage and excess risk caused by the rest ($X_t = 0$) stage.

Letting $\Delta y \to 0$ and applying Taylor expansion, the second policy has to endure excess discount and risk $\frac{\lambda + e^y}{b(k_1 - k_2 D)}\Delta y + O(\Delta y)^2$, but has the advantage in the sprint session for $-\frac{1}{x_U} Z(D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D + y}\Delta y + O(\Delta y)^2$. Taking the difference and plugging $f(D)$ we defined in equation (20) in, we have

$$\frac{(\lambda + e^y)\Delta y}{b(k_1 - k_2 D)} + \frac{Z(D)}{x_U} e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D + y}\Delta y = \frac{\Delta y}{x_U b(k_1 - k_2 D) f(D)}\left(e^y + \frac{\lambda x_U}{f(D)}\right).$$

Hence we need to discuss the sign of $f(D)$. When $f(D) \geq 0$, we always have the first policy dominates the second one. When $f(D) < 0$, there is a curve $y = \ln\left(-\frac{\lambda x_U}{f(D)}\right) = \Gamma(D)$ such that, above it, policy 2 is better than policy 1; below it, policy 1 is better than policy 2.

We construct the surge curve $g_1(D)$:

$$g_1(D) = \begin{cases} \Gamma(D) & \text{, for } 1 < D \leq D_c, \\ b(x_U - k_1)\frac{\ln D_c - \ln D}{x_U} + \frac{bk_2}{x_U}(D_c - D) + g_1(D_c) & \text{, for } D_c < D < \frac{k_1}{k_2}, \end{cases}$$

where D_c is defined as in Lemma 9. We will prove that $g_1(D)$ complies with the statement of Lemma 1. Note that, suppose the initial point is (D, y), then the expression of its trajectory can be written as

$$Y_t(D_t) = -b(x_U - k_1)\frac{\ln D_t}{x_U} - \frac{bk_2}{x_U}D_t + b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D + y$$

Note that, we always have $g_1'(D_t) \leq Y_t'(D_t) < 0$ by Lemma 9 (b). Suppose $y \leq g_1(D)$, then along the whole trajectory we will have $Y_t \leq g_1(D_t)$, i.e. we will always have $Y_t \leq \Gamma(D_t)$, which indicates it is optimal to surge (i.e. always select $X_t = x_U$).

Meanwhile, if $y > g_1(D)$,then the trajectory will interfere the area above $\Gamma(D)$, i.e. there must exist t such that $Y_t > \Gamma(D_t)$. Note that at such (D_t, Y_t), rest then sprint is strictly better than sprint. Thus it is always suboptimal to surge. □

B. *Proof of Lemma 2.*

Proof of Lemma 2. Consider the following two policies (see Figure 2):

- Policy SR: First select $X_t = x_U$ until difficulty level decreases by ΔD, and then select $X_t = 0$ until stress level decreases by Δy.
- Policy RS: First select $X_t = 0$ until stress level decreases by Δy, and then select $X_t = x_U$ until difficulty level decreases by ΔD.

Suppose the payoffs are P_{SR} and P_{RS}, respectively, and the individual's initial level is (D, y). We first compute the following time duration:

- Duration of sprint session in policy SR: $t^h_{SR} = -\frac{1}{x_U}\ln\left(\frac{D-\Delta D}{D}\right)$.
- Duration of rest session in policy SR: $t^v_{SR} = \frac{\Delta y}{b(k_1 - k_2(D-\Delta D))}$.
- Duration of sprint session in policy RS: $t^h_{RS} = -\frac{1}{x_U}\ln\left(\frac{D-\Delta D}{D}\right)$.
- Duration of rest session in policy RS: $t^v_{RS} = \frac{\Delta y}{b(k_1 - k_2 D)}$.

Letting $\Delta D \to 0$ and $\Delta y \to 0$ and applying Taylor expansion, we have

$$t_{RS}^{v} = \frac{\Delta y}{b(k_1 - k_2 D)}$$

$$t_{SR}^{v} = \frac{\Delta y}{b(k_1 - k_2 D)} - \frac{k_2 \Delta y}{b(k_1 - k_2 D)^2}\Delta D + O(\Delta D^2 \Delta y)$$

$$t_{RS}^{h} = t_{SR}^{h} = \frac{\Delta D}{D x_U} + O(\Delta D)^2$$

For policy SR, we define the stress level right after the sprint session as Y'. Take policy RS as our standard. The time advantage of policy SR is

$$-\lambda(t_{SR}^{v} - t_{RS}^{v}) = \frac{\lambda k_2}{b(k_1 - k_2 D)^2}\Delta D \Delta y + O(\Delta D^2 \Delta y),$$

while the excess risk of policy SR is

$$e^{y} t_{SR}^{h} + e^{Y'} t_{SR}^{v} - e^{y} t_{RS}^{v} - e^{y-\Delta y} t_{RS}^{h} = e^{y}\frac{bk_1 - (b+1)k_2 D}{Db(k_1 - k_2 D)^2}\Delta D \Delta y + o(\Delta D \Delta y).$$

We need to discuss the sign of $bk_1 - k_2(b+1)D$. If $bk_1 - k_2(b+1)D < 0$, we always have $P_{SR} > P_{RS}$. For $bk_1 - k_2(b+1)D > 0$, we can compute

$$e^{y}\frac{bk_1 - (b+1)k_2 D}{Db(k_1 - k_2 D)^2}\Delta D \Delta y = \frac{\lambda k_2}{b(k_1 - k_2 D)^2}\Delta D \Delta y,$$

which yields

$$y = \ln\left(\frac{\lambda k_2 D}{bk_1 - (b+1)k_2 D}\right).$$

Thus, we have the exchangeability curve $g_2(D) = \ln\left(\frac{\lambda k_2 D}{bk_1 - k_2(b+1)D}\right)$. When $y < g_2(D)$, $P_{SR} > P_{RS}$; when $y > g_2(D)$, $P_{SR} < P_{RS}$. $\square$

C. *Proof of Lemma 3.*

Proof of Lemma 3. Lemma 3 holds directly as a consequence of Theorem 1, and we will prove Theorem 1 in the following section. $\square$

APPENDIX C: PROOF OF THEOREM 1.

In this section, We prove Theorem 1 through a series of Lemmas. We first introduce an important function $\tilde{g}_2(D)$ (defined in (29)) and show that this function satisfies certain properties in Lemma 10. With this result, Lemma 11 derives, by first principle, the value function associated with the proposed policy. Then, Lemma 12 shows that the value function is C^1 and Lemma 13 shows it solves the HJB equation (9). These Lemmas, together, complete the proof of Theorem 1.

We first define

$$\tilde{g}_2(D) = \ln\left(\frac{\lambda k_2 D}{bk_1 - k_2(b+1)D}\right) + b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D, \tag{29}$$

which has the following properties.

LEMMA 10: *The function* $\tilde{g}_2 : [D_c, \frac{bk_1}{(b+1)k_2}) \to [g_1(D_c) + b(x_U - k_1)\frac{\ln D_c}{x_U} + \frac{bk_2}{x_U}D_c, \infty)$ *is a bijection.*

Proof of Lemma 10.

$$\tilde{g}_2'(D) = -\frac{bk_1}{D(k_2(b+1)D - bk_1)} + \frac{b(x_U - k_1)}{Dx_U} + \frac{bk_2}{x_U} > 0. \tag{30}$$

$\tilde{g}_2(D)$ is continuously monotonically increasing as $D \in [D_c, \frac{bk_1}{(b+1)k_2})$, so it is a bijection. □

To facilitate the presentation of subsequent results, we define several additional functions. In particular, define

$$Y_c(D) = -b(x_U - k_1)\frac{\ln D}{x_U} - \frac{bk_2}{x_U}D + b(x_U - k_1)\frac{\ln D_c}{x_U} + \frac{bk_2}{x_U}D_c + g_1(D_c),$$

and note that $Y_c(D_c) = g_1(D_c)$. Moreover, define

$$\gamma(w) = \tilde{g}_2^{-1}(w),$$

for any $w \geq g_1(D_c) + b(x_U - k_1)\frac{\ln D_c}{x_U} + \frac{bk_2}{x_U}D_c$. Note that $\gamma(w)$ is well-defined by Lemma 10. Furthermore, define

$$Q_1(D) = \int_{D_c}^{D} \left(\frac{\lambda g_2(s) + e^{g_2(s)}}{b(k_1 - k_2 s)^2} k_2 - \frac{\lambda + e^{g_2(s)}}{s(k_1 - k_2 s)} \right) ds - \frac{\lambda \ln D_c}{x_U} + \frac{\lambda g_1(D_c) - \lambda}{b(k_1 - k_2 D_c)},$$

and

$$Q_2(D, y) = -\int_{g_2(D_c) + b(x_U - k_1)\frac{\ln D_c}{x_U} + \frac{bk_2}{x_U}D_c}^{y + b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D} \frac{Z(\gamma(w))}{x_U} e^w + \frac{\lambda}{b(k_1 - k_2\gamma(w))} + \frac{\exp(w - b(x_U - k_1)\frac{\ln \gamma(w)}{x_U} - \frac{bk_2}{x_U}\gamma(w))}{b(k_1 - k_2\gamma(w))} dw.$$

LEMMA 11: *For Theorem 1's proposed policy, the corresponding value function is*

$$V(D, y) = \begin{cases} \mathcal{R} \exp\left(\frac{Z(D)}{x_U} e^{y + b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D} - \frac{\lambda \ln D}{x_U} \right) & , \textit{if } (D, y) \in \mathcal{M}_1 \\ \mathcal{R} \exp\left(-\frac{\lambda \ln D}{x_U} + \frac{\lambda g_1(D) - \lambda - \lambda y - e^y}{b(k_1 - k_2 D)} \right) & , \textit{if } (D, y) \in \mathcal{M}_{2m} \\ \mathcal{R} \exp\left(Q_1(D) - \frac{\lambda y + e^y}{b(k_1 - k_2 D)} \right) & , \textit{if } (D, y) \in \mathcal{M}_{2a} \\ \mathcal{R} \exp\left(Q_2(D, y) + \frac{Z(D)}{x_U} e^{y + b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D} - \frac{\lambda \ln D}{x_U} \right) & , \textit{if } (D, y) \in \mathcal{M}_3 \end{cases}$$

Proof of Lemma 11. If $(D, y) \in \mathcal{M}_1$, we have

$$V(D, y) = \mathcal{R} \exp\left(-\int_0^{\frac{\ln D}{x_U}} e^{Y_t} + \lambda \, dt \right)$$

Note that $dD = -X_t D_t dt$ and $Y_t(D_t) = -b(x_U - k_1)\frac{\ln D_t}{x_U} - \frac{bk_2}{x_U}D_t + b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D + y$, so we apply the change of variable and obtained that

$$\begin{aligned} V(D, y) &= \mathcal{R} \exp\left(\int_1^D -\frac{1}{s x_U} e^{-b(x_U - k_1)\frac{\ln s}{x_U} - \frac{bk_2}{x_U}s + b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D + y} ds - \int_0^{\frac{\ln D}{x_U}} \lambda \, dt \right) \\ &= \mathcal{R} \exp\left(\frac{Z(D)}{x_U} e^{y + b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U}D} - \frac{\lambda \ln D}{x_U} \right) \end{aligned}$$

If $(D,y) \in \mathcal{M}_{2m}$, we have

$$V(D,y) = V(D,g_1(D)) \cdot \exp\left(-\int_0^{\frac{y-g_1(D)}{b(k_1-k_2D)}} e^{y-b(k_1-k_2D)t} + \lambda\, dt\right)$$
$$= \mathcal{R}\exp\left(-\frac{\lambda \ln D}{x_U} + \frac{\lambda g_1(D) - \lambda - \lambda y - e^y}{b(k_1-k_2D)}\right)$$

If $(D,y) \in \mathcal{M}_{2a}$, we first compute the duration of Phase 2. Moving along with the exchangeability curve with initial level $(D, g_2(D))$, we have:

$$D_t = \frac{\frac{bk_1}{b+1}}{k_2 - (k_2 - \frac{bk_1}{D(b+1)})e^{\frac{bk_1}{b+1}t}}$$

and the duration of Phase 2 t_2 lasts for

$$t_2 = \frac{b+1}{bk_1}\ln\left(\frac{k_2 - \frac{bk_1}{D_c(b+1)}}{k_2 - \frac{bk_1}{D(b+1)}}\right),$$

and the resulting value function is

$$V(D,y) = V(D_c, g_1(D_c)) \cdot \exp\left(\int_0^{\frac{y-g_2(D)}{b(k_1-k_2D)}} \lambda - e^{y-b(k_1-k_2D)t}\, dt - \int_{D_c}^{D} \frac{e^{g_2(s)}}{sx}\, ds - \lambda t_2\right)$$
$$= \mathcal{R}\exp\left(-\frac{\lambda + e^{g_2(D_c)}}{b(k_1-k_2D_c)} - \frac{\lambda \ln D_c}{x_U} + \frac{\lambda g_2(D) + e^{g_2(D)} - \lambda y - e^y}{b(k_1-k_2D)}\right.$$
$$\left. - \int_{D_c}^{D} \frac{e^{g_2(s)}}{s(\frac{bk_1}{b+1} - k_2 s)}\, ds - \lambda t_2\right).$$

Note that we have

$$t_2 = \frac{b+1}{bk_1}\ln\left(\frac{k_2 - \frac{bk_1}{D_c(b+1)}}{k_2 - \frac{bk_1}{D(b+1)}}\right) = \frac{b+1}{bk_1}\int_{D_c}^{D} \frac{\frac{bk_1}{b+1}}{s\left(\frac{bk_1}{b+1} - k_2 s\right)}\, ds.$$

Hence we can compute that

$$V(D,y)=\mathcal{R}\exp\left(-\frac{\lambda+e^{g_2(D_c)}}{b(k_1-k_2D_c)}-\frac{\lambda\ln D_c}{x_U}+\frac{\lambda g_2(D)+e^{g_2(D)}-\lambda y-e^y}{b(k_1-k_2D)}\right.$$
$$\left.-\int_{D_c}^{D}\frac{\lambda+e^{g_2(s)}}{s(\frac{bk_1}{b+1}-k_2s)}ds\right).$$

Note that

$$\begin{aligned}&\frac{d}{dD}\left(\frac{\lambda g_2(D)+e^{g_2(D)}}{b(k_1-k_2D)}-\int_{D_c}^{D}\frac{\lambda+e^{g_2(s)}}{s(\frac{bk_1}{b+1}-k_2s)}ds\right)\\&=\frac{\lambda g_2(D)+e^{g_2(D)}}{b(k_1-k_2D)^2}k_2+\frac{\frac{k_1}{k_1-k_2D}-(b+1)}{D(bk_1-(b+1)k_2D)}(\lambda+e^{g_2(D)})\\&=\frac{\lambda g_2(D)+e^{g_2(D)}}{b(k_1-k_2D)^2}k_2-\frac{\lambda+e^{g_2(D)}}{D(k_1-k_2D)}\\&=Q_1'(D),\end{aligned}$$

and

$$-\frac{\lambda+e^{g_2(D_c)}}{b(k_1-k_2D_c)}-\frac{\lambda\ln D_c}{x_U}+\frac{\lambda g_2(D_c)+e^{g_2(D_c)}}{b(k_1-k_2D_c)}=-\frac{\lambda\ln D_c}{x_U}+\frac{\lambda g_1(D_c)-\lambda}{b(k_1-k_2D_c)}=Q_1(D_c).$$

By the fundamental theorem of Calculus, we have

$$V(D,y)=\mathcal{R}\exp\left(Q_1(D)-\frac{\lambda y+e^y}{b(k_1-k_2D)}\right).$$

If $(D,y)\in\mathcal{M}_3$, we assume $u=y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D$, then $\gamma(u)$ is the horizontal coordinate of the intersection point of $g_2(D)$ and the trajectory. The value function is

$$V(D,y)=\mathcal{R}\exp\left(-\int_1^{D_c}\frac{\lambda+e^{Y_c(s)}}{sx_U}ds-\int_{\gamma(u)}^{D}\frac{\lambda+e^{Y(s)}}{sx_U}ds-\int_{D_c}^{\gamma(u)}\frac{\lambda+e^{g_2(s)}}{s(\frac{bk_1}{b+1}-k_2s)}ds\right)$$

We first consider phase 1 and phase 3. Note that

$$-\int_1^{D_c} \frac{\lambda + e^{Y_c(s)}}{s x_U}\, ds = \frac{Z(D_c)}{x_U} e^{g_2(D_c)+b(x_U-k_1)\frac{\ln D_c}{x_U}+\frac{bk_2}{x_U}D_c} - \frac{\lambda \ln D_c}{x_U}$$

and

$$\begin{aligned}-\int_{\gamma(u)}^{D} \frac{\lambda + e^{Y(s)}}{s x_U}\, ds &= -\int_1^{D} \frac{\lambda + e^{Y(s)}}{s x_U}\, ds + \int_1^{\gamma(u)} \frac{\lambda + e^{Y(s)}}{s x_U}\, ds \\ &= \frac{Z(D)}{x_U} e^{y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} - \frac{\lambda \ln D}{x_U} \\ &\quad - \frac{Z(\gamma(u))}{x_U} e^{g_2(\gamma(u))+b(x_U-k_1)\frac{\ln \gamma(u)}{x_U}+\frac{bk_2}{x_U}\gamma(u)} + \frac{\lambda \ln \gamma(u)}{x_U},\end{aligned}$$

so $-\int_1^{D_c} \frac{\lambda+e^{Y_c(s)}}{s x_U}\, ds - \int_{\gamma(u)}^{D} \frac{\lambda+e^{Y(s)}}{s x_U}\, ds$ can be converted to

$$\frac{Z(D)}{x_U} e^{y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} - \frac{\lambda \ln D}{x_U} + \left(\frac{Z(s)}{x_U} e^{g_2(s)+b(x_U-k_1)\frac{\ln s}{x_U}+\frac{bk_2}{x_U}s} - \frac{\lambda \ln s}{x_U} \right) \Big|_{s=\gamma(u)}^{D_c}.$$

By taking derivative, we can convert $\left(\frac{Z(s)}{x_U} e^{g_2(s)+b(x_U-k_1)\frac{\ln s}{x_U}+\frac{bk_2}{x_U}s} - \frac{\lambda \ln s}{x_U} \right) \Big|_{s=\gamma(u)}^{D_c}$ into the following integral form:

$$\int_{D_c}^{\gamma(u)} \frac{\lambda + e^{g_2(s)}}{s x_U} - \left(\frac{(b+1)(k_1-k_2 s)}{bk_1 - k_2(b+1)s} - \frac{k_1 - k2s}{x_U} \right) \frac{bZ(s)}{s x_U} e^{g_2(s)+b(x_U-k_1)\frac{\ln s}{x_U}+\frac{bk_2}{x_U}s}\, ds$$

Hence we can use a change of variable by letting $w = g_2(s) + b(x_U - k_1)\frac{\ln s}{x_U} + \frac{bk_2}{x_U}s$, and therefore $dw = \left(\frac{bk_1}{s(bk_1 - k_2(b+1)s)} + \frac{b(x_U-k_1)}{s x_U} + \frac{bk_2}{x_U} \right) ds$, $s = \gamma(w)$, which yields

$$\frac{Z(s)}{x_U} e^{g_2(s)+b(x_U-k_1)\frac{\ln s}{x_U}+\frac{bk_2}{x_U}s} - \frac{\lambda \ln s}{x_U} \Big|_{s=\gamma(u)}^{D_c} = -\int_{\tilde{g}_2(D_c)}^{y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} \frac{Z(\gamma(w))}{x_U} e^w\, dw$$

Finally, for phase 2, we have

$$\int_{D_c}^{\gamma(u)} \frac{\lambda + e^{g_2(s)}}{s x_U} - \frac{\lambda + e^{g_2(s)}}{s(\dfrac{bk_1}{b+1} - k_2 s)}\, ds = \int_{D_c}^{\gamma(u)} (\lambda + e^{g_2(s)}) \frac{bk_1 - k_2(b+1)s - (b+1)x_U}{s x_U (bk_1 - k_2(b+1)s)} ds.$$

Since we can do the following transformation

$$\frac{bk_1 - k_2(b+1)s - (b+1)x_U}{s x_U (bk_1 - k_2(b+1)s)} b(k_1 - k_2 s) = \frac{dw}{ds},$$

we can transform $\int_{D_c}^{\gamma(u)} \frac{\lambda+e^{g_2(s)}}{sx_U} - \frac{\lambda+e^{g_2(s)}}{s(\frac{bk_1}{b+1}-k_2 s)} ds$ as

$$-\int_{g_2(D_c)+b(x_U-k_1)\frac{\ln D_c}{x_U}+\frac{bk_2}{x_U}D_c}^{y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} \frac{\lambda+\exp(w-b(x_U-k_1)\frac{\ln\gamma(w)}{x_U}-\frac{bk_2}{x_U}\gamma(w))}{b(k_1-k_2\gamma(w))} dw.$$

Thus,

$$\begin{aligned} V(D,y) &= \mathcal{R}\exp\left(-\int_1^{D_c} \frac{\lambda+e^{Y_c(s)}}{sx_U} ds - \int_{\gamma(u)}^{D} \frac{\lambda+e^{Y(s)}}{sx_U} ds - \int_{D_c}^{\gamma(u)} \frac{\lambda+e^{g_2(s)}}{s(\frac{bk_1}{b+1}-k_2 s)} ds\right) \\ &= \mathcal{R}\exp\left(Q_2(D,y) + \frac{Z(D)}{x_U} e^{y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} - \frac{\lambda \ln D}{x_U}\right) \quad \square \end{aligned}$$

LEMMA 12: *$V(D,y)$ defined in Lemma 11 is C^1.*

Proof of Lemma 12. It suffices to show that $V(D,y)$ is C^1 at the region boundaries. Also, since all four pieces of $V(D,y)$ are exponential functions, it suffices to show the exponential parts are C^1. For simplicity, we denote the exponential parts of $V(D,y)$ in $\mathcal{M}_1$, $\mathcal{M}_{2m}$, $\mathcal{M}_{2a}$, and $\mathcal{M}_3$ by V_1, V_2, V_3, and V_4 respectively, i.e.,

$$\begin{aligned} V_1(D,y) &= \frac{Z(D)}{x_U} e^{y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} - \frac{\lambda \ln D}{x_U}, \\ V_2(D,y) &= -\frac{\lambda \ln D}{x_U} + \frac{\lambda g_1(D) - \lambda - \lambda y - e^y}{b(k_1-k_2 D)}, \\ V_3(D,y) &= Q_1(D) - \frac{\lambda y + e^y}{b(k_1-k_2 D)}, \\ V_4(D,y) &= Q_2(D,y) + \frac{Z(D)}{x_U} e^{y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} - \frac{\lambda \ln D}{x_U}. \end{aligned}$$

On the boundary of $\mathcal{M}_1$ and $\mathcal{M}_{2m}$, i.e. the set of (D,y) such that $y = g_1(D) = \Gamma(D)$, note that

$$e^{g_1(D)+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} = -\frac{x_U(\lambda+e^{g_1(D)})}{bZ(D)(k_1-k_2 D)}.$$

By the result of Lemma 11, a few steps of direct computation lead to

$$\frac{\partial}{\partial D}(V_1 - V_2)\big|_{y=g_1(D)} = 0,$$

and

$$\frac{\partial}{\partial y}(V_1 - V_2)\big|_{y=g_1(D)} = \frac{Z(D)}{x_U} e^{g_1(D)+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D} + \frac{\lambda + e^{g_1(D)}}{b(k_1-k_2D)} = 0,$$

so we have $V(D,y)$ is C^1 on this boundary.

On the boundary of $\mathcal{M}_{2m}$ and $\mathcal{M}_{2a}$, i.e. the set of (D,y) such that $D = D_c$, note that we have $g_1(D_c) = g_2(D_c)$.

$$\begin{aligned}\frac{\partial}{\partial D}(V_2 - V_3)\big|_{D=D_c} &= -\frac{\lambda + e^{g_1(D_c)}}{D_c x_U} - \frac{\lambda + e^{g_1(D_c)}}{b(k_1-k_2D_c)}(\frac{b(x_U-k_1)}{D_c x_U} + \frac{bk_2}{x_U}) \\ &\quad + \frac{\lambda + e^{g_2(D_c)}}{D_c(k_1-k_2D_c)} \\ &= 0,\end{aligned}$$

and

$$\frac{\partial}{\partial y}(V_2 - V_3)\big|_{D=D_c} = -\frac{\lambda + e^y}{b(k_1-k_2D_c)} + \frac{\lambda + e^y}{b(k_1-k_2D_c)} = 0,$$

so we have $V(D,y)$ is C^1 on this boundary.

On the boundary of $\mathcal{M}_{2a}$ and $\mathcal{M}_3$, i.e. the set of (D,y) such that $y = g_2(D)$, we have

$$\begin{aligned}\frac{\partial}{\partial D}(V_3 - V_4)\big|_{y=g_2(D)} &= -\frac{\lambda + e^{g_2(D)}}{D(k_1-k_2D)} + \frac{\lambda + e^{g_2(D)}}{b(k_1-k_2D)}(\frac{b(x_U-k_1)}{Dx_U} + \frac{bk_2}{x_U}) + \frac{e^{g_2(D)}+\lambda}{Dx_U} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}(V_3 - V_4)\big|_{y=g_2(D)} &= -\frac{\lambda + e^{g_2(D)}}{b(k_1-k_2D)} + \frac{\lambda + e^{g_2(D)}}{b(k_1-k_2D)} \\ &= 0,\end{aligned}$$

so we have $V(D,y)$ is C^1 on this boundary.

On the boundary of $\mathcal{M}_1$ and $\mathcal{M}_3$, i.e. the set of (D,y) such that $y = g_1(D) = Y_c(D)$, we have

$$\frac{\partial}{\partial D}(V_1 - V_4)\big|_{y=Y_c(D)} = \left(\frac{\lambda + e^{g_1(D_c)}}{b(k_1-k_2D_c)} - \frac{\lambda + e^{g_1(D_c)}}{b(k_1-k_2D_c)}\right)\left(\frac{b(x_U-k_1)}{Dx_U} + \frac{bk_2}{x_U}\right) = 0.$$

The equation holds because $Y_c(D)+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D=g_1(D_c)+b(x_U-k_1)\frac{\ln D_c}{x_U}+\frac{bk_2}{x_U}D_c$. Also,

$$\frac{\partial}{\partial y}(V_1-V_4)\big|_{y=Y_c(D)}=-\frac{\lambda+e^{g_1(D_c)}}{b(k_1-k_2D_c)}+\frac{\lambda+e^{g_1(D_c)}}{b(k_1-k_2D_c)}=0.$$

so we have $V(D,y)$ is C^1 on this boundary, which completes the proof of $V(D,y)$ is C^1. □

Now we prove that $V(D,y)$ satisfies the HJB equation.

LEMMA 13: $V(D,y)$ *solves the HJB equation* (9).

Proof of Lemma 13. Note that, $-D\frac{\partial V(D,y)}{\partial D}+b\frac{\partial V(D,y)}{\partial y}$ has the same sign with $-D\frac{\partial V_i}{\partial D}+b\frac{\partial V_i}{\partial y}$ for $i=1,\dots,4$ and it suffices to show $e^y+\lambda=-DX_t\frac{\partial V_i}{\partial D}+b(X_t-(k_1-k_2D))\frac{\partial V_i}{\partial y}$ for $i=1,\dots,4$.

In the first region $\mathcal{M}_1$, note that we always have $y\le g_1(D)$. Thus we have

$$-D\frac{\partial V_1}{\partial D}+b\frac{\partial V_1}{\partial y}=\frac{\lambda+e^y}{x_U}+b(k_1-k_2D)\frac{Z(D)}{x_U^2}e^{y+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D}$$

If $x_U+bZ(D)(k_1-k_2D)e^{b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D}<0$, we have:

$$\begin{aligned}-D\frac{\partial V_1}{\partial D}+b\frac{\partial V_1}{\partial y}&=\frac{\lambda}{x_U}+\frac{e^y}{x_Ue^{g_1(D)}}\left(e^{g_1(D)}+b(k_1-k_2D)\frac{Z(D)}{x_U}e^{g_1(D)+\frac{b(x_U-k_1)\ln D}{x_U}+\frac{bk_2D}{x_U}}\right)\\&=\frac{\lambda}{x_U}\left(1-\frac{e^y}{e^{g_1(D)}}\right)\\&\ge 0.\end{aligned}$$

where the second equal sign holds by $e^{g_1(D)+b(x_U-k_1)\frac{\ln D}{x_U}+\frac{bk_2}{x_U}D}=-\frac{x_U(\lambda+e^{g_1(D)})}{bZ(D)(k_1-k_2D)}$.

Otherwise, we have

$$-D\frac{\partial V_1}{\partial D}+b\frac{\partial V_1}{\partial y}\ge\frac{\lambda+e^y}{x_U}-\frac{e^y}{x_U}=\frac{\lambda}{x_U}\ge 0.$$

Hence, we can compute

$$-Dx_U\frac{\partial V_1}{\partial D}+b(x_U-(k_1-k_2D))\frac{\partial V_1}{\partial y}=-Dx_U\left(\frac{e^y}{-Dx_U}-\frac{\lambda}{Dx_U}\right)=e^y+\lambda$$

In the second region $\mathcal{M}_{2m}$, suppose that

$$f_2(y) = (k_2(b+1)D - bk_1)(e^y - e^{g_1(D)}) + \lambda k_2 D(y - g_1(D))$$

and

$$f_2'(y) = (k_2(b+1)D - bk_1)e^y + \lambda k_2 D.$$

Note that we have $y > g_1(D)$ and $k_2(b+1)D - bk_1 < 0$, so we can compute that

$$\begin{aligned} f_2'(y) &< (k_2(b+1)D - bk_1)e^{g_1(D)} + \lambda k_2 D \\ &= \frac{x_U + k_2 D Z(D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U} D}}{x_U + bZ(D)(k_1 - k_2 D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U} D}} \lambda b(k_1 - k_2 D) \end{aligned}$$

Recall that D_c has the property that

$$x_U + k_2 D_c Z(D_c) e^{b(x_U - k_1)\frac{\ln D_c}{x_U} + \frac{bk_2}{x_U} D_c} = 0.$$

Region $\mathcal{M}_{2m}$ indicates $D \le D_c$, so we have

$$x_U + k_2 D Z(D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U} D} \ge 0.$$

Hence we have

$$\frac{x_U + k_2 D Z(D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U} D}}{x_U + bZ(D)(k_1 - k_2 D) e^{b(x_U - k_1)\frac{\ln D}{x_U} + \frac{bk_2}{x_U} D}} \lambda b(k_1 - k_2 D) \le 0,$$

and therefore,

$$f_2'(y) < 0$$

for $y > g_1(D)$ and

$$f_2(y) < f_2(g_1(D)) = 0.$$

Hence we obtain that

$$-D\frac{\partial V_2}{\partial D} + b\frac{\partial V_2}{\partial y} < 0.$$

Hence we have

$$-b(k_1 - k_2 D)\frac{\partial V_2}{\partial y} = \lambda + e^y$$

In $\mathcal{M}_{2a}$, we have

$$-D\frac{\partial V_3}{\partial D} + b\frac{\partial V_3}{\partial y} = \frac{(k_2(b+1)D - bk_1)(e^y - e^{g_2(D)}) + \lambda k_2 D(y - g_2(D))}{b(k_1 - k_2 D)^2}$$

and obviously $b(k_1 - k_2 D)^2 > 0$. Suppose that

$$f_3(y) = (k_2(b+1)D - bk_1)(e^y - e^{g_2(D)}) + \lambda k_2 D(y - g_2(D))$$

and

$$f_3'(y) = (k_2(b+1)D - bk_1)e^y + \lambda k_2 D.$$

Note that we have $y \geq g_2(D)$ and $k_2(b+1)D - bk_1 < 0$, so we can compute that

$$f_3'(y) \leq (k_2(b+1)D - bk_1)e^{g_2(D)} + \lambda k_2 D = 0.$$

Obviously that $f_3(g_2(D)) = 0$, so we have

$$f_3(y) \leq f_3(g_2(D)) = 0$$

and therefore

$$-D\frac{\partial V_3}{\partial D} + b\frac{\partial V_3}{\partial y} \leq 0.$$

Hence we have

$$-b(k_1 - k_2 D)\frac{\partial V_3}{\partial y} = \lambda + e^y$$

For the region $\mathcal{M}_3$, we introduce $w = y + \frac{b(x_U - k_1)}{x_U}\ln D + \frac{bk_2}{x_U}D$ for our convenience. In the fourth region, we have

$$-D\frac{\partial V_4}{\partial D} + b\frac{\partial V_4}{\partial y} = -\frac{b(k_1 - k_2 D)}{x_U}\left(\frac{Z(\gamma(w))}{x_U}e^w - \frac{Z(D)}{x_U}e^w\right)$$

$$-\frac{b(k_1-k_2D)}{x_U}\frac{\lambda+\exp\left(w-b(x_U-k_1)\frac{\ln\gamma(w)}{x_U}-\frac{bk_2}{x_U}\gamma(w)\right)}{b(k_1-k_2\gamma(w))}$$
$$-D\left(\frac{Z'(D)}{x_U}e^w-\frac{\lambda}{Dx_U}\right)$$

Suppose that

$$f_4(D,w)=-D\frac{\partial V_4}{\partial D}+b\frac{\partial V_4}{\partial y}$$

Note that, although at first, w is a function of D and y, we can now view w and D are independent variables, with the constraint $D>\gamma(w)$. It is like we project the original region from the (D,y)-plane to a (D,w)-plane. This transformation will benefit us in learning the lower bound of $-D\frac{\partial V_4}{\partial D}+b\frac{\partial V_4}{\partial y}$.

Note that we have

$$f_4(\gamma(w),w)=0,$$

so we compute

$$\frac{\partial f_4}{\partial D}=\frac{1}{Dx_U(k_1-k_2\gamma(w))}\Bigg(\lambda k_2D+k_2D\exp(w-b(x_U-k_1)\frac{\ln\gamma(w)}{x_U}-\frac{bk_2}{x_U}\gamma(w))$$
$$-b(k_1-k_2\gamma(w))e^{w-b(x_U-k_1)\frac{\ln D}{x_U}-\frac{bk_2}{x_U}D}\Bigg)+\frac{bk_2}{x_U}\left(\frac{Z(\gamma(w))}{x_U}e^w-\frac{Z(D)}{x_U}e^w\right)$$

Note that $D>\gamma(w)$, so we have $\frac{Z(\gamma(w))}{x_U}e^w-\frac{Z(D)}{x_U}e^w>0$ and

$$-b(k_1-k_2\gamma(w))e^{w-b(x_U-k_1)\frac{\ln D}{x_U}-\frac{bk_2}{x_U}D}>-b(k_1-k_2\gamma(w))e^{w-b(x_U-k_1)\frac{\ln\gamma(w)}{x_U}-\frac{bk_2}{x_U}\gamma(w)}$$

and therefore,

$$\begin{aligned}&\lambda k_2D+k_2D\exp\left(w-b(x_U-k_1)\frac{\ln\gamma(w)}{x_U}-\frac{bk_2}{x_U}\gamma(w)\right)-b(k_1-k_2\gamma(w))e^y\\ >&\lambda k_2\gamma(w)+(k_2\gamma(w)-b(k_1-k_2\gamma(w)))\exp\left(w-b(x_U-k_1)\frac{\ln\gamma(w)}{x_U}-\frac{bk_2}{x_U}\gamma(w)\right)\\ =&0\end{aligned}$$

Hence we always have $\frac{\partial f_4}{\partial D} > 0$ when $D > \gamma(w)$, so we know

$$f_4(D,w) > f_4(\gamma(w),w) = 0,$$

and therefore,

$$-D\frac{\partial V_4}{\partial D} + b\frac{\partial V_4}{\partial y} > 0.$$

and

$$-Dx_U\frac{\partial V_4}{\partial D} + b(x_U - (k_1 - k_2 D))\frac{\partial V_4}{\partial y} = -Dx_U\left(\frac{Z'(D)}{x_U}e^w - \frac{\lambda}{Dx_U}\right) = e^y + \lambda \quad \square$$

Proof of Theorem 1. Theorem 1 holds as a result of Lemma 11, Lemma 12, and Lemma 13. $\square$

APPENDIX D: RESULTS UNDER GENERAL STRESS FUNCTION

In this section, we release our assumption on the stress intensity function to a general function $h(y)$. In subsection A, we derive the generalized exchangeability curve, and in subsection B, we establish the generalized surge curve. Based on the non-negativity, monotonicity, and log-convexity of $h(y)$, we derive a series of important properties in Lemma 14, which is essential to the construction of surge curve and generalized optimal policy. In subsection C, we prove Lemma 4, 5, 6 and Theorem 2.

A. Derivation of the Generalized Exchageability Curve

Consider the following two policies:

- Policy SR: First select $X_t = x_U$ until difficulty level decreases by ΔD, and then select $X_t = 0$ until stress level decreases by Δy.
- Policy RS: First select $X_t = 0$ until stress level decreases by Δy, and then select $X_t = x_U$ until difficulty level decreases by ΔD.

Suppose the payoffs are P_{SR} and P_{RS}, respectively, and the individual's initial level is (D, y). We first compute the following time duration:

- Duration of sprint session in policy SR: $t^h_{SR} = -\frac{1}{x_U}\ln\left(\frac{D-\Delta D}{D}\right)$.
- Duration of rest session in policy SR: $t^v_{SR} = \frac{\Delta y}{b(k_1 - k_2(D-\Delta D))}$.

- Duration of sprint session in policy RS: $t^h_{RS} = -\frac{1}{x_U}\ln\left(\frac{D-\Delta D}{D}\right)$.
- Duration of rest session in policy RS: $t^v_{RS} = \frac{\Delta y}{b(k_1-k_2D)}$.

Letting $\Delta D \to 0$ and $\Delta y \to 0$ and applying Taylor expansion, we have

$$
\begin{aligned}
t^v_{RS} &= \frac{\Delta y}{b(k_1-k_2D)} \\
t^v_{SR} &= \frac{\Delta y}{b(k_1-k_2D)} - \frac{k_2\Delta y}{b(k_1-k_2D)^2}\Delta D + o(\Delta D\Delta y) \\
t^h_{RS} &= t^h_{SR} = \frac{\Delta D}{Dx_U} + o(\Delta D)
\end{aligned}
$$

For policy SR, we define the stress level right after the sprint session as Y'. Take policy RS as our standard. The time advantage of policy SR is

$$
-\lambda(t^v_{SR} - t^v_{RS}) = \frac{\lambda k_2}{b(k_1-k_2D)^2}\Delta D\Delta y + o(\Delta D\Delta y),
$$

while the excess risk of policy SR is

$$
\begin{aligned}
& h(y)t^h_{SR} + h(Y')t^v_{SR} - h(y)t^v_{RS} - h(y-\Delta y)t^h_{RS} \\
= & -\frac{h(y)k_2}{b(k_1-k_2D)^2}\Delta D\Delta y + h'(y)\left(\frac{1}{Dx_U} + \frac{b(x_U-(k_1-k_2D))}{Dx_Ub(k_1-k_2D)}\right)\Delta D\Delta y + o(\Delta D\Delta y) \\
= & -\frac{h(y)k_2}{b(k_1-k_2D)^2}\Delta D\Delta y + \frac{h'(y)}{D(k_1-k_2D)}\Delta D\Delta y + o(\Delta D\Delta y).
\end{aligned}
$$

Hence, if (D, y) is on the exchangeability curve, we should have

$$
-\frac{h(y)k_2}{b(k_1-k_2D)^2}\Delta D\Delta y + \frac{h'(y)}{D(k_1-k_2D)}\Delta D\Delta y = \frac{\lambda k_2}{b(k_1-k_2D)^2}\Delta D\Delta y,
$$

which yields

$$
-k_2Dh(y) + b(k_1-k_2D)h'(y) = \lambda k_2 D. \tag{31}
$$

Note that, a few steps of transformations of equation (31) lead to

$$
D = \frac{bk_1h'(y)}{\lambda k_2 + bk_2h'(y) + k_2h(y)}. \tag{32}
$$

This indicates that the exchangeability curve is not necessarily a function with respect to D (in other words, for a fixed level of D, there might be multiple corresponding y that make equation (31) hold).

B. Derivation of the Generalized Surge Curve

We first derive $\tilde{g}_1^{(h)}(\cdot)$, which is essential to the derivation of the surge curve. We consider the following two strategies:

- Policy 1: Always select $X_t = x_U$.
- Policy 2: First select $X_t = 0$ and let Y_t decrease by Δy, then always select $X_t = x_U$.

Suppose the initial point is (D, y) and we take policy 1 as our standard. Then the strength of policy 2 is the relatively lower risk in the sprint ($X_t = x_U$) stage, but it suffers from the time disadvantage and excess risk caused by the rest ($X_t = 0$) stage.

Letting $\Delta y \to 0$ and applying Taylor expansion, the second policy has a weakness

$$\int_0^{\frac{\Delta y}{b(k_1 - k_2 D)}} \lambda + h(y - b(k_1 - k_2 D)t)dt = \frac{\lambda + h(y)}{b(k_1 - k_2 D)}\Delta y + o(\Delta y),$$

but has the advantage

$$\begin{aligned} &\int_1^D \frac{1}{s x_U}\left[h\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D - s) + y\right)\right. \\ &\qquad\qquad \left. - h\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D - s) + y - \Delta y\right)\right] ds \\ = \;& \Delta y \int_1^D \frac{1}{s x_U} h'\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D - s) + y\right) ds + o(\Delta y). \end{aligned}$$

Hence, if (D, y) is on the surge curve, we should have

$$\frac{\lambda + h(y)}{b(k_1 - k_2 D)} - \int_1^D \frac{1}{s x_U} h'\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D - s) + y\right) ds = 0. \quad (33)$$

Note that, if we view equation (33) as an equation with respect to D, we have the following findings: when $D = 1$, the left-hand side of equation (33) is always positive for any level of y; when $D \to (k_1/k_2)^-$, then the left-hand side of equation (33) is also positive for any level of y. Thus, to make equation (33) have any root, it suffices to make

$$\frac{\lambda + h(y)}{b(k_1 - k_2 D)} - \int_1^D \frac{1}{s x_U} h'\left(b(x_U - k_1) \frac{\ln D - \ln s}{x_U} + \frac{b k_2}{x_U}(D - s) + y \right) ds < 0,$$

for some (D, y). This can be realized by making b sufficiently large (c.f. Lemma 7 in the case of $h(y) = e^y$), because when holding all other parameters constant, as we increase b, we will have $\frac{\lambda + h(y)}{b(k_1 - k_2 D)}$ decrease while $\int_1^D \frac{1}{s x_U} h'\left(b(x_U - k_1) \frac{\ln D - \ln s}{x_U} + \frac{b k_2}{x_U}(D - s) + y \right) ds$ increase. Intuitively, this means we have the trade-off between sprinting and resting only when the individual is sensitive enough. Suppose $b \to 0$, then the individual hardly accumulates any stress when sprinting, and the stress barely reduces when resting. In this way, surging is always optimal for the individual, which makes our problem trivial. Therefore, in the remainder of the proof, we always assume b is large enough.

Then we show the following important result.

LEMMA 14: *Suppose $\ln h(y)$ is convex and b is sufficiently large, then the followings hold:*

i). For any $D \in [1, k_1/k_2)$, there exists at most one y that satisfies (35), i.e. (35) becomes a continuous function $y = \tilde{g}_1^{(h)}(D)$.

ii). For any $D \in [1, k_1/k_2)$, there exists at most one y that satisfies (31), i.e. (31) becomes a continuous function $y = g_2^{(h)}(D)$. Moreover, $g_2(\cdot)$ is monotonically increasing.

iii). The two curves $y = \tilde{g}_1^{(h)}(D)$ and $y = g_2^{(h)}(D)$ intersect at exactly one point (D_c, y_c), and we have $\tilde{g}_1^{(h)\,\prime}(D_c) = -\frac{b(x_U - k_1)}{D_c x_U} - \frac{b k_2}{x_U}$.

Moreover, define

$$Y_c(D) = -b(x_U - k_1)\frac{\ln D}{x_U} - \frac{b k_2}{x_U} D + b(x_U - k_1)\frac{\ln D_c}{x_U} + \frac{b k_2}{x_U} D_c + g_1(D_c) \tag{34}$$

to be the trajectory path passing through (D_c, y_c). Then the followings hold.

iv). For $D < D_c$, $\tilde{g}_1^{(h)\,\prime}(D) < Y_c'(D) = -\frac{b(x_U - k_1)}{D x_U} - \frac{b k_2}{x_U}$.

v). For $D > D_c$, $Y_c(D) < \tilde{g}_1^{(h)}(D) < g_2^{(h)}(D)$.

Proof of Lemma 14. To prove (i), we transform equation (33) as follows:

$$\frac{\lambda + h(y)}{bh(y)(k_1 - k_2 D)} - \int_1^D \frac{1}{s x_U} \frac{h'\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D-s) + y\right)}{h(y)} ds = 0, \tag{35}$$

and we define the left-hand side of equation (35) as a function:

$$H_1(y) = \frac{\lambda + h(y)}{bh(y)(k_1 - k_2 D)} - \int_1^D \frac{1}{s x_U} \frac{h'\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D-s) + y\right)}{h(y)} ds. \tag{36}$$

Here, we hold D fixed and let y be the only variable. We show that $H_1(y)$ has at most one zero by proving $H_1(y)$ is monotone. Note that

$$H_1(y) = \frac{1}{h(y)} \left(\frac{\lambda + h(y)}{b(k_1 - k_2 D)} - \int_1^D \frac{h'\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D-s) + y\right)}{s x_U} ds \right)$$

Since $\lim_{y \to -\infty} h(y) = 0$ and $\lim_{y \to -\infty} h'(y) = 0$, we have

$$\lim_{y \to -\infty} H_1(y) > 0$$

We study the three terms of $H_1(y)$, respectively. The first term of $H_1(y)$, $\frac{\lambda}{bh(y)(k_1 - k_2 D)}$, monotonically decreases when y increases because $h'(y) > 0$. The second term, $\frac{1}{b(k_1 - k_2 D)}$ is a constant term with respect to y.

To study the third term, we define

$$w = b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D - s)$$

where $s \in [1, D]$. It is easy to show that $w \geq 0$: taking derivative with respect to s, we have

$$\frac{dw}{ds} = -\frac{b(x_U - (k_1 - k_2 s))}{s x_U} < 0$$

Let $s = D$, then corresponding $w = 0$. Hence, we have $w \geq 0$ for all $s \in [1, D]$. We can study the monotonicity of the third term by $h'(w + y)/h(y)$. Take derivative of $h'(w +$

$y)/h(y)$ with respect to y, we have

$$\frac{d}{dy}\left(\frac{h'(w+y)}{h(y)}\right) = \frac{h''(w+y)h(y) - h'(w+y)h'(y)}{[h(y)]^2}$$

Since $\ln h(y)$ is convex, which indicates its first order derivative $h'(y)/h(y)$ is non-decreasing, and its second order derivative is non-negative, i.e.,

$$h''(y)h(y) - [h'(y)]^2 \geq 0.$$

Thus, we have $h''(y) \geq [h'(y)]^2/h(y)$, which leads to

$$h''(w+y) \geq \frac{[h'(w+y)]^2}{h(w+y)} \geq \frac{h'(w+y)h'(y)}{h(y)}$$

for $w \geq 0$, where the last inequality holds because $h'(y)/h(y)$ is non-decreasing. This implies $\frac{d}{dy}\left(\frac{h'(w+y)}{h(y)}\right) \geq 0$ and therefore, $-\int_1^D \frac{1}{sx_U} \frac{h'\left(b(x_U-k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D-s)+y\right)}{h(y)} ds$ is non-increasing as y increases. Thus, $H_1(y)$ monotonically decreases as y increases, so $H_1(y)$ has at most one zero as we fix D. Hence, for any $D \in [1, k_1/k_2)$, there is at most one corresponding y that solves equation (35), and we denote such function as $y = \tilde{g}_1^{(h)}(D)$.

Then we prove (ii). Note that, equation (31) can be transformed as equation (32), which is the function of D with respect to y. Therefore, to make $y = g_2^{(h)}(D)$ well-defined and monotone, it suffices to show that in equation (32), D is monotone to y.

Define $H_2(y) = \frac{bk_1h'(y)}{\lambda k_2 + bk_2h'(y) + k_2h(y)}$. We have

$$\begin{aligned} H_2'(y) &= \frac{bk_1h''(y)k_2(\lambda + bh'(y) + h(y)) - bk_1h'(y)(bk_2h''(y) + k_2h'(y))}{(\lambda k_2 + bk_2h'(y) + k_2h(y))^2} \\ &= \frac{bk_1k_2[h''(y)(\lambda + h(y)) - [h'(y)]^2]}{(\lambda k_2 + bk_2h'(y) + k_2h(y))^2} \\ &\geq \frac{bk_1k_2(h''(y)\lambda)}{(\lambda k_2 + bk_2h'(y) + k_2h(y))^2} \end{aligned}$$

Therefore, $H_2(y)$ is monotone. Hence, there exists a function $y = g_2^{(h)}(D)$ that increases with D.

Then we prove (iii). We first briefly study the behavior of $y = g_1^{(h)}(D)$. When b is large enough, $y = g_1^{(h)}(D)$ has two asymptotes, namely $D = D_L$ and $D = D_U$ where $1 \leq D_L <$

$D_U \le k_1/k_2$. This is because when $D \to 1^+$, the left-hand side of equation (33) is always positive for any level of y; when $D \to (k_1/k_2)^-$, then the left-hand side of equation (33) is also positive for any level of y. Also note that, as $D \to D_L^+$ or $D \to D_U^-$, we have $y \to +\infty$. This is because, by the monotonicity of function $H_1(y)$ defined in equation (36), the left-hand side of equation (35) decreases as y increases.

Take derivative of the implicit function $y = \tilde{g}_1^{(h)}(D)$ defined in equation (33), we have

$$\frac{h'(y)y'}{b(k_1 - k_2 D)} + \frac{k_2(\lambda + h(y))}{b(k_1 - k_2 D)^2} - \frac{h'(y)}{Dx_U} - \left(\frac{b(x_U - k_1)}{Dx_U} + \frac{bk_2}{x_U} + y'\right)$$
$$\cdot \int_1^D \frac{1}{sx_U} h''\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D - s) + y\right) ds = 0. \qquad (37)$$

We first show the existence of the intersection point. When b is sufficiently large, then $y = \tilde{g}_1^{(h)}(D)$ has two asymptotes $D = D_L$ and $D = D_U$, so we have $\tilde{g}_1^{(h)\prime}(D) \in (-\infty, \infty)$ when $D \in (D_L, D_U)$, and there must exist some point D_c such that $\tilde{g}_1^{(h)\prime}(D_c) = -\frac{b(x_U - k_1)}{D_c x_U} - \frac{bk_2}{x_U}$. Substituting $y' = -\frac{b(x_U - k_1)}{D_c x_U} - \frac{bk_2}{x_U}$, $D = D_c$, and $y = \tilde{g}_1^{(h)}(D_c) = y_c$ to equation (37), we have

$$\frac{h'(y_c)}{b(k_1 - k_2 D_c)}\left(-\frac{b(x_U - k_1)}{D_c x_U} - \frac{bk_2}{x_U}\right) + \frac{k_2(\lambda + h(y_c))}{b(k_1 - k_2 D_c)^2} - \frac{h'(y_c)}{D_c x_U} = 0.$$

A few steps of transformation leads to

$$-k_2 D_c h(y_c) + b(k_1 - k_2 D_c)h'(y_c) = \lambda k_2 D_c,$$

which complies with equation (31), i.e., (D_c, y_c) is on the curve $y = g_2^{(h)}(D)$.

Then we prove the uniqueness of such intersection points. Since (D_c, y_c) is on both $y = \tilde{g}_1^{(h)}(D)$ and $y = g_2^{(h)}(D)$, by a few steps of transformation on equation (31), we have

$$h'(y_c) = \frac{\lambda k_2 D_c + k_2 D_c h(y_c)}{b(k_1 - k_2 D_c)}. \qquad (38)$$

Plugging (38) in (37), a few steps of transformation gives

$$\frac{k_2 D_c(\lambda + h(y_c))}{b^2(k_1 - k_2 D_c)^2}\left(\tilde{g}_1^{(h)\prime}(D_c) + \frac{b(x_U - k_1)}{D_c x_U} + \frac{bk_2}{x_U}\right) - \left(\frac{b(x_U - k_1)}{D_c x_U} + \frac{bk_2}{x_U} + \tilde{g}_1^{(h)\prime}(D_c)\right)$$
$$\cdot \int_1^{D_c} \frac{1}{sx_U} h''\left(b(x_U - k_1)\frac{\ln D_c - \ln s}{x_U} + \frac{bk_2}{x_U}(D_c - s) + y_c\right) ds = 0,$$

which indicates we have

$$\tilde{g}_1^{(h)\,\prime}(D_c) = -\frac{b(x_U - k_1)}{D_c x_U} - \frac{bk_2}{x_U},$$

or

$$\frac{(\lambda + h(y_c))k_2 D_c}{b^2(k_1 - k_2 D_c)^2} = \int_1^{D_c} \frac{1}{s x_U} h'' \left(b(x_U - k_1)\frac{\ln D_c - \ln s}{x_U} + \frac{bk_2}{x_U}(D_c - s) + y \right) ds. \tag{39}$$

We will show that equation (39) cannot hold. Suppose by contradiction that we have equation (39) holds. Since (D_c, y_c) is on $y = \tilde{g}_1^{(h)}(D)$, equation (33) gives

$$\frac{\lambda + h(y_c)}{b(k_1 - k_2 D_c)} = \int_1^{D_c} \frac{1}{s x_U} h' \left(b(x_U - k_1)\frac{\ln D_c - \ln s}{x_U} + \frac{bk_2}{x_U}(D_c - s) + y \right) ds. \tag{40}$$

By equation (39) and (40), we should have

$$\begin{aligned}\int_1^{D_c} \frac{1}{s} \Bigg[\frac{k_2 D_c}{b(k_1 - k_2 D_c)} h' &\left(b(x_U - k_1)\frac{\ln D_c - \ln s}{x_U} + \frac{bk_2}{x_U}(D_c - s) + y_c \right) \\ -h'' &\left(b(x_U - k_1)\frac{\ln D_c - \ln s}{x_U} + \frac{bk_2}{x_U}(D_c - s) + y_c \right) \Bigg] ds = 0. \end{aligned} \tag{41}$$

Recall equation (31), which is the implicit function of the exchangeability curve. Take derivative on both sides with respect to D, we have

$$-k_2 h(y) - k_2 D h'(y) y' - bk_2 h'(y) + b(k_1 - k_2 D) h''(y) y' = \lambda k_2,$$

which leads to

$$\big(b(k_1 - k_2 D) h''(y) - k_2 D h'(y) \big) y' = \lambda k_2 + k_2 h(y) + bk_2 h'(y). \tag{42}$$

Since in our setting, $y = g_2^{(h)}(D)$ monotonically increases, i.e. $y' > 0$, we have $b(k_1 - k_2 D)h''(y) - k_2 D h'(y) > 0$, or equivalently,

$$\frac{h''(y)}{h'(y)} > \frac{k_2 D}{b(k_1 - k_2 D)}.$$

Since $y = g_2^{(h)}(D)$, we have $D = g_2^{(h)\ -1}(y)$ because $y = g_2^{(h)}(D)$ is monotone. Then for any $w \geq 0$, we have

$$\frac{h''(w+y_c)}{h'(w+y_c)} > \frac{k_2 g_2^{(h)\ -1}(w+y_c)}{b(k_1 - k_2 g_2^{(h)\ -1}(w+y_c))} \geq \frac{k_2 D_c}{b(k_1 - k_2 D_c)}$$

where the last inequality holds because $g_2^{(h)\ -1}(w+y_c) \geq D_c$ when $w \geq 0$. Therefore, we have $\frac{k_2 D_c}{b(k_1 - k_2 D_c)} h'(w+y_c) - h''(w+y_c) < 0$ for $w \geq 0$, which indicates equation (41) cannot hold. Hence, at (D_c, y_c), we must have $g_1^{(h)\ \prime}(D_c) = -\frac{b(x_U - k_1)}{D_c x_U} - \frac{bk_2}{x_U}$.

The uniqueness is because we always have $\tilde{g}_1^{(h)\ \prime}(D_c) < 0$ and $g_2^{(h)\ \prime}(D_c) > 0$ at the intersection point (D_c, y_c), so the intersection point is unique.

We then prove (iv). By (i) and (iii), we know that $\tilde{g}_1^{(h)}(D) > g_2(D)$ for $D < D_c$, and $\tilde{g}_1^{(h)}(D) < g_2(D)$ when $D > D_c$. We consider the points (D, y) which are on the curve $y = \tilde{g}_1^{(h)}(D)$. If $D < D_c$, we have

$$D < \frac{bk_1 h'(y)}{\lambda k_2 + bk_2 h'(y) + k_2 h(y)},$$

which indicates

$$\frac{h'(y)}{D(k_1 - k_2 D)} > \frac{k_2(\lambda + h(y))}{b(k_1 - k_2 D)^2}.$$

Suppose $\frac{h'(y)}{D(k_1 - k_2 D)} = \frac{k_2(\lambda + h(y))}{b(k_1 - k_2 D)^2} + C_1$ for some $C_1 > 0$. Substituting $\frac{k_2(\lambda + h(y))}{b(k_1 - k_2 D)^2} = \frac{h'(y)}{D(k_1 - k_2 D)} - C_1$ to equation (37), we have

$$\left(\frac{y'}{b(k_1 - k_2 D)} - \frac{1}{D x_U}\right) h'(y) + \frac{h'(y)}{D(k_1 - k_2 D)} - C_1 - \left(\frac{b(x_U - k_1)}{D x_U} + \frac{bk_2}{x_U} + y'\right)$$
$$\cdot \int_1^D \frac{1}{s x_U} h''\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D - s) + y\right) ds = 0,$$

i.e.,

$$\left(\frac{y'}{b(k_1 - k_2 D)} + \frac{x_U - (k_1 - k_2 D)}{D x_U (k_1 - k_2 D)}\right) h'(y) - C_1 - \left(\frac{b(x_U - k_1)}{D x_U} + \frac{bk_2}{x_U} + y'\right)$$
$$\cdot \int_1^D \frac{1}{s x_U} h''\left(b(x_U - k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D - s) + y\right) ds = 0. \quad (43)$$

We view y' as the only variable of equation (43), and we know that when taking $y' = -\frac{b(x_U-k_1)}{Dx_U} - \frac{bk_2}{x_U}$, the left-hand side of equation (43) becomes $-C_1$. Also note that, we have

$$\frac{h'(y)}{b(k_1-k_2D)} - \int_1^D \frac{1}{sx_U} h''\left(b(x_U-k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D-s)+y\right) ds < 0, \quad (44)$$

for $y = \tilde{g}_1(D)$. This is because, recall $H_1(y)$ defined in equation (36), we have $H_1(y) > 0$ when $y < \tilde{g}_1^{(h)}(D)$, and $H_1(y) < 0$ when $y > \tilde{g}_1^{(h)}(D)$. Note that

$$h(y)H_1(y) = \frac{\lambda + h(y)}{b(k_1-k_2D)} - \int_1^D \frac{1}{sx_U} h'\left(b(x_U-k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D-s)+y\right) ds$$

Since $h(y) > 0$, we have $h(y)H_1(y) > 0$ when $y < \tilde{g}_1(D)$, and $h(y)H_1(y) < 0$ when $y > \tilde{g}_1(D)$. The left-hand side of inequality (44) is just $\frac{d}{dy}(h(y)H_1(y))$, so we must have $\frac{d}{dy}(h(y)H_1(y)) < 0$ at $y = \tilde{g}_1^{(h)}(D)$. Thus, we have inequality (44) holds. Therefore, to make equation (43) holds, we have $y' < -\frac{b(x_U-k_1)}{Dx_U} - \frac{bk_2}{x_U}$, i.e. $\tilde{g}_1^{(h)}{}'(D) < -\frac{b(x_U-k_1)}{Dx_U} - \frac{bk_2}{x_U}$ for all $D < D_c$.

Finally, we prove (v). We have $g_2^{(h)}(D) > \tilde{g}_1^{(h)}(D)$ as a direct consequence by (iii): $y = g_2(D)$ and $y = \tilde{g}_1(D)$ has a unique intersection point (D_c, y_c) where $g_2^{(h)}{}'(D_c) > \tilde{g}_1^{(h)}{}'(D_c)$. To prove $\tilde{g}_1^{(h)}(D) > Y_c(D)$, we take a similar approach as (iv). We consider the (D, y) such that $y = \tilde{g}_1^{(h)}(D)$. If $D > D_c$, we have

$$D > \frac{bk_1h'(y)}{\lambda k_2 + bk_2h'(y) + k_2h(y)},$$

which indicates

$$\frac{h'(y)}{D(k_1-k_2D)} < \frac{k_2(\lambda+h(y))}{b(k_1-k_2D)^2},$$

Suppose $\frac{h'(y)}{D(k_1-k_2D)} = \frac{k_2(\lambda+h(y))}{b(k_1-k_2D)^2} - C_2$ for some $C_2 > 0$. Substituting $\frac{k_2(\lambda+h(y))}{b(k_1-k_2D)^2} = \frac{h'(y)}{D(k_1-k_2D)} - C_2$ to equation (37), we have

$$\begin{aligned}&\left(\frac{y'}{b(k_1-k_2D)} - \frac{1}{Dx_U}\right)h'(y) + \frac{h'(y)}{D(k_1-k_2D)} + C_2 - \left(\frac{b(x_U-k_1)}{Dx_U} + \frac{bk_2}{x_U} + y'\right)\\ &\cdot \int_1^D \frac{1}{sx_U} h''\left(b(x_U-k_1)\frac{\ln D - \ln s}{x_U} + \frac{bk_2}{x_U}(D-s)+y\right) ds = 0,\end{aligned}$$

i.e.,

$$\left(\frac{y'}{b(k_1-k_2D)}+\frac{x_U-(k_1-k_2D)}{Dx_U(k_1-k_2D)}\right)h'(y)+C_2-\left(\frac{b(x_U-k_1)}{Dx_U}+\frac{bk_2}{x_U}+y'\right)$$
$$\cdot\int_1^D\frac{1}{sx_U}h''\left(b(x_U-k_1)\frac{\ln D-\ln s}{x_U}+\frac{bk_2}{x_U}(D-s)+y\right)ds=0. \tag{45}$$

We view y' as the only variable of equation (45), and we know that when taking $y'=-\frac{b(x_U-k_1)}{Dx_U}-\frac{bk_2}{x_U}$, the left-hand side of equation (45) becomes C_2. Also, recall that, by inequality (44), we then have $y'>-\frac{b(x_U-k_1)}{Dx_U}-\frac{bk_2}{x_U}$, which leads to $\tilde{g}_1^{(h)}(D)>Y_c(D)$. $\square$

In the condition of Lemma 14, we define the surge curve as $y=g_1^{(h)}(D)$, where

$$g_1^{(h)}(D)=\begin{cases}\tilde{g}_1^{(h)}(D) & \text{, if } D\le D_c,\\ Y_c(D) & \text{, if } D>D_c,\end{cases}$$

and the exchangeability curve as $y=g_2^{(h)}(D)$.

C. *Proof of Lemma 4, 5, 6, and Theorem 2*

Proof of Lemma 4. We consider the points in the set $\{(D,y);y\le g_1^{(h)}(D)\}$. By Lemma 14, we can conclude $g_1^{(h)}{}'(D)\le-\frac{b(x_U-k_1)}{Dx_U}-\frac{bk_2}{x_U}<0$. This means for any trajectory whose start point (D,y) is below the surge curve, its whole trajectory path stays below the surge curve. Hence, by the derivation of the surge curve, stopping at any point along its trajectory path is suboptimal. Therefore, we have the "surge region" defined as $\{(D,y);y\le g_1^{(h)}(D)\}$. $\square$

Proof of Lemma 5. Lemma 5 is a direct consequence of our previous derivation of the generalized exchangeability curve. For simplicity, we use the expression defined in equation (32). Above the exchangeability curve, we have

$$D<\frac{bk_1h'(y)}{\lambda k_2+bk_2h'(y)+k_2h(y)},$$

and a few steps of transformation give

$$-\frac{h(y)k_2}{b(k_1-k_2D)^2}+\frac{h'(y)}{D(k_1-k_2D)}>\frac{\lambda k_2}{b(k_1-k_2D)^2},$$

where the left-hand side is the excess risk of SR and the right-hand side is the time advantage of SR. Hence, above the exchangeability curve, we have $P_{RS} > P_{SR}$. Similarly, we have $P_{RS} < P_{SR}$ below the exchangeability curve. □

Proof of Lemma 6. We prove Lemma 6 by contradiction, which is similar to our proof of Lemma 3. Suppose there exists $(D, y) \in \mathcal{M}_2^{(h)}$ such that, selecting $X_t = x_U$ is optimal. Then, by Lemma 5, it is never optimal for the individual to shift from x_U to 0 above the exchangeability curve. However, by Lemma 14, the exchangeability curve increases in D while the trajectory decreases in D. As the process proceeds (i.e., D_t decreases as t increases), the individual's status (D_t, Y_t) will remain above the exchangeability curve, which never lets the individual shift from x_U to any other status, and leads to a "surge" strategy, i.e. always choosing $X_t = x_U$. However, contradiction arises since (D, y) is not in the surge region, i.e. surge is suboptimal at (D, y). Therefore, there cannot exist $(D, y) \in \mathcal{M}_2^{(h)}$ such that selecting $X_t = x_U$ is optimal.

Then suppose by contradiction that, there exists $(D, y) \in \mathcal{M}_3^{(h)}$ such that selecting $X_t = 0$ is optimal. By Lemma 5, it is never optimal for the individual to shift from 0 to x_U below the exchangeability curve. However, as long as the individual keeps resting, the individual's status (D_t, Y_t) will remain below the exchangeability curve, which will result in a zero value because D_t never decreases. This is obviously suboptimal. Therefore, there cannot exist $(D, y) \in \mathcal{M}_3^{(h)}$ such that selecting $X_t = 0$ is optimal. □

Proof of Theorem 2. If $(D, y) \in \mathcal{M}_1^{(h)}$, which is the surge region. Hence, by Lemma 4, we always select $X_t = x_U$.

If $(D, y) \in \mathcal{M}_{2m}^{(h)}$, then by Lemma 6, the individual selects $X_t = 0$ when $(D_t, Y_t) \in \mathcal{M}_2^{(h)}$. By equation (4) and letting $X_t = 0$, this phase will last for $t = \frac{y - g_1^{(h)}(D)}{b(k_1 - k_2 D)}$ and (D_t, Y_t) intersects with $y = \tilde{g}_1^{(h)}(D)$ and breaks into the surge region. As the trajectory breaks into the surge region, the individual shifts to $X_t = x_U$. By Lemma 14 property (iv), when the individual begins to sprint, it will never intersect with $y = \tilde{g}_1^{(h)}(D)$ again, and therefore never shifts away from $X_t = x_U$.

If $(D, y) \in \mathcal{M}_{2a}^{(h)}$, by Lemma 6, the individual selects $X_t = 0$ when $(D_t, Y_t) \in \mathcal{M}_2^{(h)}$. Then, by Lemma 6, the trajectory shall coincide with the exchangeability curve $y =$

$g_2^{(h)}(D)$, i.e., we shall have $\frac{dY_t}{dD_t} = g_2^{(h)}{}'(D)$, which is

$$-\frac{b(X_t - k_1)}{D_t X_t} - \frac{bk_2}{X_t} = \frac{\lambda k_2 + k_2 h(Y_t) + bk_2 h'(Y_t)}{b(k_1 - k_2 D_t)h''(Y_t) - k_2 D_t h'(Y_t)},$$

where the right-hand side is derived from equation (42). Therefore, we have

$$X_t = \frac{b(k_1 - k_2 D_t)[b(k_1 - k_2 D_t)h''(Y_t) - k_2 D_t h'(Y_t)]}{b^2(k_1 - k_2 D_t)h''(Y_t) + [\lambda + h(Y_t)]k_2 D_t}.$$

This phase will last until (D_t, Y_t) reaches (D_c, y_c), i.e. reaches the surge region. The individual will then always select $X_t = x_U$.

If $(D, y) \in \mathcal{M}_3^{(h)}$, by Lemma 6, the individual selects $X_t = x_U$ when $(D_t, Y_t) \in \mathcal{M}_2^{(h)}$. Then its trajectory is very similar to our previous analysis. The individual will select $X_t = \frac{b(k_1 - k_2 D_t)[b(k_1 - k_2 D_t)h''(Y_t) - k_2 D_t h'(Y_t)]}{b^2(k_1 - k_2 D_t)h''(Y_t) + [\lambda + h(Y_t)]k_2 D_t}$ until (D_t, Y_t) reaches (D_c, y_c), and finally always select $X_t = x_U$. $\square$